CO327 (2021Spring) Assignment 4

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Introduction to assignment 4 The goal of this assignment is to familiarize yourself with calling library / solver of the MATLAB programming language.

The solvers we will use

- Linear programming: the code `x = linprog(f,A,b)` will solve

$$\min_{\mathbf{x}} \ \mathbf{f}^\top \mathbf{x} \ \text{ s.t. } \ \mathbf{A}\mathbf{x} \leq \mathbf{b},$$

  for $\mathbf{f} \in \mathbb{R}^n$ (an n-by-1 column vector), $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m \times n}$ (a m-by-n matrix) and $\mathbf{b} \in \mathbb{R}^m$.

- Linear Integer programming: the code `x = intlinprog(f,I,A,b)` will solve

$$\min_{\mathbf{x}} \ \mathbf{f}^\top \mathbf{x} \ \text{ s.t. } \ \mathbf{A}\mathbf{x} \leq \mathbf{b}, \ \mathbf{x}_I \in \mathbb{N}$$

  for $\mathbf{f} \in \mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and I is a subset of $\{1, 2, \ldots, n\}$ labelling which element of $\mathbf{x}$ has to be an integer.

- Quadratic program with linear inequality constraint: the code `x = quadprog(H,f,A,b)` will solve

$$\min_{\mathbf{x}} \ \frac{1}{2}\mathbf{x}^\top \mathbf{H}\mathbf{x} + \mathbf{f}^\top \mathbf{x} \ \text{ s.t. } \ \mathbf{A}\mathbf{x} \leq \mathbf{b},$$

  for $\mathbf{H} \in \mathbb{R}^{n \times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{f} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

Remark

- Your problem in the original form may not have the same expression as the one used in these solvers, so your first task is to convert your problem into the form used by the solver. Beware of the max and min, and the direction of the inequality sign.

- You can also study these codes and make use of their advanced structure. For example, the code `x = linprog(f,A,b,Aeq,beq,l,b)` will solve

$$\min_{\mathbf{x}} \ \mathbf{f}^\top \mathbf{x} \ \text{ s.t. } \ \mathbf{A}\mathbf{x} \leq \mathbf{b}, \ \mathbf{A}_{eq}\mathbf{x} = \mathbf{b}_{eq}, \ \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}.$$

  where $\mathbf{A}_{eq}, \mathbf{b}_{eq}$ refer to the constants for the equality constraints and $\mathbf{l}, \mathbf{u}$ are lower bound and upper bounds. For empty input, the square bracket [ ] can be used. For example, the code `x = linprog(f,[],[],Aeq,beq,[],u)` will solve

$$\min_{\mathbf{x}} \ \mathbf{f}^\top \mathbf{x} \ \text{ s.t. } \ \mathbf{A}_{eq}\mathbf{x} = \mathbf{b}_{eq}, \ \mathbf{x} \leq \mathbf{u}.$$

- Make sure the variables you defined in your code have the correct dimension. Be-careful of transpose (apostrophe in MATLAB), do not mix up column vector and row vector.

Submission of assignment You submit two things: the MAATLAB m-files and a pdf.

- The MATLAB m-files contain the code that solve the problem. The m-file should contains all the variable (those $\mathbf{f}, \mathbf{A}, \mathbf{b}$, etc discussed above). The m-file has to be "run-able", has no bug. You get points when the m-file can be run in MATLAB and show the correct solution.
- The pdf contains the answer of the questions, plus explanation of the your code (if you think extra explanation is needed).

Zip your mfiles and pdf into one single .7z (or .zip) file with the file name:

```
your student ID+''underscore''+ a4
```

and submit to the dropbox in WATERLOO LEARN.

Submission deadline: 23:55 (Eastern Daylight Time) 30/June/2021

1 Simple problems (5 points)

1. Find $\mathbf{x} = [x_1, x_2]^\top$ that maximizes $7x_1 + 5x_2$, subject to $2x_1 + x_2 \leq 100$, $4x_1 + 3x_2 \leq 240$, $x_1 \geq 0$, $x_2 \geq 0$. If the problem has no solution, explain why.
2. Find $\mathbf{x} = [x_1, x_2, x_3]^\top$ that minimizes $-5x_1 - 4x_2 - 6x_3$, subject to $x_1 - x_2 + x_3 \leq 20$, $-3x_1 - 2x_2 - 4x_3 \geq -42$, $3x_1 + 2x_2 \leq 30$, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$. If the problem has no solution, explain why.
3. Find $\mathbf{x} = [x_1, x_2]^\top$ that maximizes $150x_1 + 175x_2$ subject to $7x_1 + 11x_2 \leq 77$, $10x_1 + 8x_2 \leq 80$, $0 \leq x_1 \leq 9$, $0 \leq x_2 \leq 6$. If the problem has no solution, explain why.
4. Find $\mathbf{x} = [x_1, x_2, x_3]^\top$ that minimizes $\dfrac{1}{6}\mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{p}^\top \mathbf{x}$ with $\mathbf{Q} = \begin{bmatrix} 11 & 7 & 3 \\ 7 & 5 & 1 \\ 3 & 1 & 9 \end{bmatrix}$, $\underset{p}{\cancel{\mathbf{q}}} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$. If the problem has no solution, explain why.

2 LP relaxation of IP (10 points)

1. Draw the feasible region of the following constraints: $x_1 + x_2 \leq 6$, $5x_1 + 9x_2 \leq 45$, $x_1 \geq 0, x_2 \geq 0$.
2. Find $\mathbf{x}_{LP} = [x_1, x_2]^\top$ that maximizes $5x_1 + 8x_2$ subject to $x_1 + x_2 \leq 6$, $5x_1 + 9x_2 \leq 45$, $x_1 \geq 0, x_2 \geq 0$. If the problem has no solution, explain why.
3. Find $\mathbf{x}_{IP} = [x_1, x_2]^\top$ that maximizes $5x_1 + 8x_2$ subject to $x_1 + x_2 \leq 6$, $5x_1 + 9x_2 \leq 45$, $x_1 \geq 0, x_2 \geq 0$ and x_1, x_2 are integers. If the problem has no solution, explain why.
4. (On LP relaxation) Round the solution $\mathbf{x}$ for the LP problem to:
   - The closest integer $\mathbf{x}_{LP}^{Round}$
   - The closest integer that is feasible for the constraint $\mathbf{x}_{LP}^{Round,feasible}$

   Let $f(\mathbf{x})$ be the objective function values at $\mathbf{x}$. Compare the following values: $f(\mathbf{x}_{LP})$, $f(\mathbf{x}_{IP})$, $f(\mathbf{x}_{LP}^{Round})$ and $f(\mathbf{x}_{LP}^{Round,feasible})$. Explain why these objective function values are different. What can you tell about the effectiveness of LP relaxation on solving this IP?

3 BIP (6 points)

1. Find $\mathbf{x} = [x_1, x_2, x_3, x_4]^\top$ that maximizes $8x_1+11x_2-6x_3+4x_4$ subject to $5x_1+7x_2-4x_3+3x_4 \leq 14$ and all $x_i \in \{0, 1\}$. If the problem has no solution, explain why.
2. Convert the previous problem into a Knapsack problem and solve the new BIP. Does the new problem share the same solution to the previous one? If no, explain why.

4 Norm minimization's (18 points)

1. Find $\mathbf{x} = [x_1, x_2]^\top$ that minimizes $\left\| \begin{bmatrix} 2x_1 + 3x_2 - 1 \\ x_2 \end{bmatrix} \right\|_\infty$. Explain how you derive the solution.
2. Consider the following optimization problem

$$\text{minimize} \|\mathbf{Ax} - \mathbf{b}\|_1 + \|\mathbf{x}\|_\infty$$

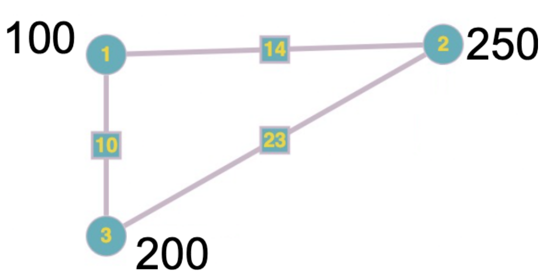
   where $\mathbf{x}$ is a n-by-1 vector, $\mathbf{A}$ is a m-by-n matrix and b is a m-by-1 vector. Here $\mathbf{x}$ is the optimization variable and $\mathbf{A}, \mathbf{b}$ are given.

   Turn this problem to an equivalent LP. Explain how you derive the solution.
3. Minimize $\|\mathbf{Ax} - \mathbf{b}\|_1 + \|\mathbf{x}\|_\infty$ for $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

5 Scheduling problem (10 points)

You want to determine their arrival time of trains. There are 10 trains such that their arrival time are arranged in ascending order $0 \leq t_1 \leq \cdots \leq t_{10}$. You want to maximize the smallest time difference between consecutive trains, subject to time range constraints $l_i \leq t_i \leq u_i$, $i = 1, 2 \ldots, 10$, where $i = l_i = u_i - 2$ for all i. Solve this optimization problem. If the problem has no solution, explain why.

6 p-median problem (20 points)



Consider the graph. It has 3 nodes $\{1, 2, 3\}$, each node represent a city. The number next to the node represents the demand of electricity of the city. The numbers on the edge represent the cost of transporting electricity between the two connected cities. Note that there is no cost to transport electricity within the same city.

This region is now deciding where to place a power plant and to decide the amount of power to supply to each city, such that the total cost of transporting the electricity is minimized. Model this problem as a 1-median problem, solve the problem. Repeat this for $p = 2$. If the problem has no solution, explain why.

(Hint: it is recommended to use the code `intlinprog(c,set,A,b,Aeq,beq,l)`)

this is ``ell'' not ``one''

END.