Solution to CO327 (2021Spring) Assignment 4

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Introduction to assignment 4 The goal of this assignment is to familiarize yourself with calling library / solver of the MATLAB programming language.

The solvers we will use

- Linear programming: the code `x = linprog(f,A,b)` will solve

$$\min_{\mathbf{x}} \ \mathbf{f}^\top \mathbf{x} \ \text{s.t.} \ \mathbf{A}\mathbf{x} \leq \mathbf{b},$$

  for $\mathbf{f} \in \mathbb{R}^n$ (an n-by-1 column vector), $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m \times n}$ (a m-by-n matrix) and $\mathbf{b} \in \mathbb{R}^m$.

- Linear Integer programming: the code `x = intlinprog(f,I,A,b)` will solve

$$\min_{\mathbf{x}} \ \mathbf{f}^\top \mathbf{x} \ \text{s.t.} \ \mathbf{A}\mathbf{x} \leq \mathbf{b}, \ \mathbf{x}_I \in \mathbb{N}$$

  for $\mathbf{f} \in \mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and I is a subset of $\{1, 2, \dots, n\}$ labelling which element of $\mathbf{x}$ has to be an integer.

- Quadratic program with linear inequality constraint: the code `x = quadprog(H,f,A,b)` will solve

$$\min_{\mathbf{x}} \ \frac{1}{2}\mathbf{x}^\top \mathbf{H}\mathbf{x} + \mathbf{f}^\top \mathbf{x} \ \text{s.t.} \ \mathbf{A}\mathbf{x} \leq \mathbf{b},$$

  for $\mathbf{H} \in \mathbb{R}^{n \times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{f} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

Remark

- Your problem in the original form may not have the same expression as the one used in these solvers, so your first task is to convert your problem into the form used by the solver. Beware of the max and min, and the direction of the inequality sign.

- You can also study these codes and make use of their advanced structure. For example, the code `x = linprog(f,A,b,Aeq,beq,l,b)` will solve

$$\min_{\mathbf{x}} \ \mathbf{f}^\top \mathbf{x} \ \text{s.t.} \ \mathbf{A}\mathbf{x} \leq \mathbf{b}, \ \mathbf{A}_{eq}\mathbf{x} = \mathbf{b}_{eq}, \ \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}.$$

  where $\mathbf{A}_{eq}, \mathbf{b}_{eq}$ refer to the constants for the equality constraints and $\mathbf{l}, \mathbf{u}$ are lower bound and upper bounds. For empty input, the square bracket [ ] can be used. For example, the code `x = linprog(f,[],[],Aeq,beq,[],u)` will solve

$$\min_{\mathbf{x}} \ \mathbf{f}^\top \mathbf{x} \ \text{s.t.} \ \ \mathbf{A}_{eq}\mathbf{x} = \mathbf{b}_{eq}, \ \mathbf{x} \leq \mathbf{u}.$$

- Make sure the variables you defined in your code have the correct dimension. Be-careful of transpose (apostrophe in MATLAB), do not mix up column vector and row vector.

Submission of assignment You submit two things: the MATLAB m-files and a pdf.

- The MATLAB m-files contain the code that solve the problem. The m-file should contains all the variable (those $\mathbf{f}, \mathbf{A}, \mathbf{b}$, etc discussed above). The m-file has to be "run-able", has no bug. You get points when the m-file can be run in MATLAB and show the correct solution.
- The pdf contains the answer of the questions, plus explanation of the your code (if you think extra explanation is needed).

Zip your mfiles and pdf into one single .7z (or .zip) file with the file name:

`your student ID+‘‘underscore’’+ a4`

and submit to the dropbox in WATERLOO LEARN.

Submission deadline: 23:55 (Eastern Daylight Time) 30/June/2021

1 Simple problems (5 points)

1. Find $\mathbf{x} = [x_1, x_2]^\top$ that maximizes $7x_1 + 5x_2$, subject to $2x_1 + x_2 \leq 100$, $4x_1 + 3x_2 \leq 240$, $x_1 \geq 0$, $x_2 \geq 0$. If the problem has no solution, explain why.
2. Find $\mathbf{x} = [x_1, x_2, x_3]^\top$ that minimizes $-5x_1 - 4x_2 - 6x_3$, subject to $x_1 - x_2 + x_3 \leq 20$, $-3x_1 - 2x_2 - 4x_3 \geq -42$, $3x_1 + 2x_2 \leq 30$, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$. If the problem has no solution, explain why.
3. Find $\mathbf{x} = [x_1, x_2]^\top$ that maximizes $150x_1 + 175x_2$ subject to $7x_1 + 11x_2 \leq 77$, $10x_1 + 8x_2 \leq 80$, $0 \leq x_1 \leq 9$, $0 \leq x_2 \leq 6$. If the problem has no solution, explain why.
4. Find $\mathbf{x} = [x_1, x_2, x_3]^\top$ that minimizes $\frac{1}{6}\mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{p}^\top \mathbf{x}$ with $\mathbf{Q} = \begin{bmatrix} 11 & 7 & 3 \\ 7 & 5 & 1 \\ 3 & 1 & 9 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$. If the problem has no solution, explain why.

Sol

1. $\mathbf{x}^* = [30, 40]^\top$.
2. $\mathbf{x}^* = [0, 15, 3]^\top$.
3. $\mathbf{x}^* = [4.8889, 3.8889]^\top$
4. $\mathbf{x}^* = [22.5, -33.75, -6.75]^\top$ (1 pt) for `quadprog(Q,p)` (1 pt) Note that $\mathbf{Q} = \frac{1}{3}\begin{bmatrix} 11 & 7 & 3 \\ 7 & 5 & 1 \\ 3 & 1 & 9 \end{bmatrix}$

   Note: QP with no constraint can still have optimal point(s). For example

   $$\min x^2,$$

   there is no constraint, but it doesn't mean the problem is unbounded. This problem is bounded below by zero (the smallest value of this quadratic is zero).

2 LP relaxation of IP (10 points)

1. Draw the feasible region of the following constraints: $x_1 + x_2 \leq 6$, $5x_1 + 9x_2 \leq 45$, $x_1 \geq 0, x_2 \geq 0$.
2. Find $\mathbf{x}_{LP} = [x_1, x_2]^\top$ that maximizes $5x_1 + 8x_2$ subject to $x_1 + x_2 \leq 6$, $5x_1 + 9x_2 \leq 45$, $x_1 \geq 0, x_2 \geq 0$. If the problem has no solution, explain why.
3. Find $\mathbf{x}_{IP} = [x_1, x_2]^\top$ that maximizes $5x_1+8x_2$ subject to $x_1+x_2 \leq 6$, $5x_1+9x_2 \leq 45$, $x_1 \geq 0, x_2 \geq 0$ and x_1, x_2 are integers. If the problem has no solution, explain why.
4. (On LP relaxation) Round the solution $\mathbf{x}$ for the LP problem to:
   - The closest integer $\mathbf{x}_{LP}^{Round}$
   - The closest integer that is feasible for the constraint $\mathbf{x}_{LP}^{Round,feasible}$

   Let $f(\mathbf{x})$ be the objective function values at $\mathbf{x}$. Compare the following values: $f(\mathbf{x}_{LP})$, $f(\mathbf{x}_{IP})$, $f(\mathbf{x}_{LP}^{Round})$ and $f(\mathbf{x}_{LP}^{Round,feasible})$. Explain why these objective function values are different. What can you tell about the effectiveness of LP relaxation on solving this IP?

Sol

1. Figure 1 point
2. $\mathbf{x}_{LP}^* = [2.25, 3.75]$ (1 pt), optimal val = -41.25 for the matlab min problem, and = +41.25 for the original problem (1 pt)
3. $\mathbf{x}_{IP}^* = [0, 5]$ (1 pt), optimal val = -40 (matlab min problem), and = +40 for the original problem (1 pt)
4. Rounded $\mathbf{x}_{LP}^* = [2, 4]$ (1 pt), objective value = +42, but such point is infeasible. Such objective value 42 is higher than 40 for the IP because the point is infeasible (i.e., a constraint is relaxed). (1 pt)

   Rounded feasible $\mathbf{x}_{LP}^* = [2, 3]$ or $[3, 3]$ (1 pt), objective value = +34 (or +39), but such feasible point is far from the optimal solution $[0, 5]$. (1 pt)

   LP relaxation does not work here as rounding will round $\mathbf{x}_{LP}$ far away from $\mathbf{x}_{IP}$. (1 pt)

3 BIP (6 points)

1. Find $\mathbf{x} = [x_1, x_2, x_3, x_4]^\top$ that maximizes $8x_1+11x_2-6x_3+4x_4$ subject to $5x_1+7x_2-4x_3+3x_4 \leq 14$ and all $x_i \in \{0, 1\}$. If the problem has no solution, explain why.
2. Convert the previous problem into a Knapsack problem and solve the new BIP. Does the new problem share the same solution to the previous one? If no, explain why.

Sol

1. Reformulate the problem to

$$\min \quad -[8, 11, -6, 4]\mathbf{x} \quad \text{s.t.} \quad [5, 7, -4, 3]\mathbf{x} \leq 14, x \in \{0, 1\}$$

   MATLAB code `intlinprog(c,[1:4],A,b,[],[],zeros(4,1),ones(4,1))` gives $\mathbf{x}^* = [1, 1, 0, 0]^\top$ (2 pts), optimal cost = -19 for the MATLAB min problem and = +19 for the original problem (2 pts).

2. Knapsack problem : maximize $8x_1 + 11x_2 + 6y_3 + 4x_4$**-6** subject to $5x_1 + 7x_2 + 4y_3 + 3x_4 \leq 18$, all $x, y \in \{0, 1\}$

   - Not the same sol, because they give different solution $\mathbf{x}$ (or similar explanation) only get 1pt
   - Same sol get all 2 pts.

   Detail explanation why same solution Students may think that the problems give different $\mathbf{x}$ so they have different solution, this is true. However, both problem share the same optimal value. For $\mathbf{x} = [1, 1, 1, 0]$ in part 2 it gives 8(1) + 11(1) + 6(1) + 4(0) **-6** = 19. The trap here is the constant term -6.

4 Norm minimization's (18 points)

1. Find $\mathbf{x} = [x_1, x_2]^\top$ that minimizes $\left\| \begin{bmatrix} 2x_1 + 3x_2 - 1 \\ x_2 \end{bmatrix} \right\|_\infty$. Explain how you derive the solution.

2. Consider the following optimization problem

$$\text{minimize} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1 + \|\mathbf{x}\|_\infty$$

   where $\mathbf{x}$ is a n-by-1 vector, $\mathbf{A}$ is a m-by-n matrix and b is a m-by-1 vector. Here $\mathbf{x}$ is the optimization variable and $\mathbf{A}, \mathbf{b}$ are given.

   Turn this problem to an equivalent LP. Explain how you derive the solution.

3. Minimize $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1 + \|\mathbf{x}\|_\infty$ for $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

Sol

1 One suggested re-formulation:

$$\min \left\| \begin{bmatrix} 2x_1 + 3x_2 - 1 \\ x_2 \end{bmatrix} \right\|_\infty = \min \ \max \Big\{ |2x_1 + 3x_2 - 1|, |x_2| \Big\}$$

Let $t \geq 0$ such that $|2x_1 + 3x_2 - 1| \leq t$ and $|x_2| \leq t$, then the equivalent problem is

$$\min\ t \ \text{s.t.}\ t \geq 0, |2x_1 + 3x_2 - 1| \leq t,\ |x_2| \leq t$$

In LP form

$$\min \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \\ t \end{bmatrix} \quad \text{s.t.} \quad \begin{bmatrix} 0 & 0 & -1 \\ 2 & 3 & -1 \\ -2 & -3 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ t \end{bmatrix} \leq \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

Sol: $x_1^* = 0.5, x_2^* = 0, t^* = 0$

If x^* correct get all 6 points

Otherwise deduct by parts: min max 1pt, t 1 pt, new LP 2 pts, sol. x 2 pts

2 For $\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m$ and $t \in \mathbb{R}$

$$\min \mathbf{1}^\top \mathbf{y} + t \quad \text{s.t.} \quad -\mathbf{y} \leq \mathbf{A}\mathbf{x} - \mathbf{b} \leq \mathbf{y}, \quad -t\mathbf{1} \leq \mathbf{x} \leq t\mathbf{1} \qquad \text{(6 pts)}$$

cost function 2 pts, each inequality constraint 2 pt

The complete LP is

$$\min_{\mathbf{x},\mathbf{y},t} \begin{bmatrix} \mathbf{0}_{n\times1} \\ \mathbf{1}_{m\times1} \\ 1 \end{bmatrix}^\top \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ t \end{bmatrix} \quad \text{s.t.} \quad \begin{bmatrix} -\mathbf{A}_{m\times n} & -\mathbf{I}_{m\times m} & \mathbf{0}_{m\times1} \\ \mathbf{A}_{m\times n} & -\mathbf{I}_{m\times m} & \mathbf{0}_{m\times1} \\ -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times m} & -\mathbf{1}_{n\times1} \\ \mathbf{I}_{n\times n} & \mathbf{0}_{n\times m} & -\mathbf{1}_{n\times1} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ t \end{bmatrix} \leq \begin{bmatrix} -\mathbf{b}_{m\times1} \\ \mathbf{b}_{m\times1} \\ \mathbf{0}_{m\times1} \\ \mathbf{0}_{m\times1} \end{bmatrix}$$

3 $\mathbf{x}^* = [-0.5, 0]^\top, \mathbf{y}^* = [0, 0, 1.5]^\top, t = 0.5$. Optimal cost $= 2$.

6 pts (2 each, if optimal cost correct get all 6 pts)

5 Scheduling problem (10 points)

You want to determine their arrival time of trains. There are 10 trains such that their arrival time are arranged in ascending order $0 \leq t_1 \leq \cdots \leq t_{10}$. You want to maximize the smallest time difference between consecutive trains, subject to time range constraints $l_i \leq t_i \leq u_i$, $i = 1, 2 \ldots, 10$, where $i = l_i = u_i - 2$ for all i. Solve this optimization problem. If the problem has no solution, explain why.

Sol

$$\max \begin{bmatrix} \mathbf{0}_{10\times1} \\ 1 \end{bmatrix}^\top \begin{bmatrix} \mathbf{t} \\ \Delta \end{bmatrix} \ \text{s.t.} \ \begin{bmatrix} \mathbf{I}_{10\times10} & \mathbf{0}_{10\times1} \\ -\mathbf{I}_{10\times10} & \mathbf{0}_{10\times1} \\ \mathbf{D}_{9\times10} & \mathbf{1}_{9\times1} \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \Delta \end{bmatrix} \leq \begin{bmatrix} \mathbf{u}_{10\times1} \\ \mathbf{l}_{10\times1} \\ \mathbf{0}_{9\times1} \end{bmatrix} \qquad \text{(5 pts)}$$

where $\mathbf{D}$ is the first-order difference operator :

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & \ddots \end{bmatrix} \qquad \text{(1 pt)}$$

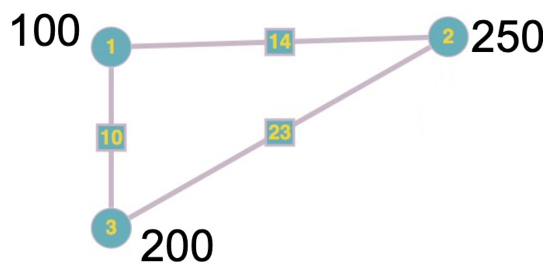
Note that this is a max problem, so we need to call `td = linprog(-c,A,b);` (1 pt), which gives

$$\texttt{t} = [1.0000,\ 2.2222,\ 3.4444,\ 4.6667,\ 5.8889,\ 7.1111,\ 8.3333,\ 9.5556,\ 10.7778,\ 12.0000]$$

with smallest time gap as 1.2222.

2 pts for **t** and 1 pts for time gap

6 p-median problem (20 points)



Consider the graph. It has 3 nodes $\{1, 2, 3\}$, each node represent a city. The number next to the node represents the demand of electricity of the city. The numbers on the edge represent the cost of transporting electricity between the two connected cities. Note that there is no cost to transport electricity within the same city.

This region is now deciding where to place a power plant and to decide the amount of power to supply to each city, such that the total cost of transporting the electricity is minimized. Model this problem as a 1-median problem, solve the problem. Repeat this for $p = 2$. If the problem has no solution, explain why.

(Hint: it is recommended to use the code `intlinprog(c,set,A,b,Aeq,beq,l)`)

Sol

Introduction Let $y_i \in \{0, 1\}$ be the decision variable of building a power plant in city i. Let $x_{ij} \in \mathbb{R}_+$ be the portion of electricity supplied to city j by power plant i. The cost c_{ij} is represented by the weights of the edges. Here we have $c_{jj} = 0$. The demand d_j is denoted by the node value.

The cost function is then

$$\begin{array}{lll} \min & 100(0)x_{11} + 100(14)x_{21} + 100(10)x_{31} & \text{all the cost to supply electricity to city 1} \\ & +250(14)x_{12} + 250(0)x_{22} + 250(23)x_{32} & \text{all the cost to supply electricity to city 2} \\ & +200(10)x_{13} + 200(23)x_{23} + 200(0)x_{33} & \text{all the cost to supply electricity to city 3} \end{array}$$

Constraints: $x_{ij} \geq 0, \quad y_i \in \{0, 1\}, \quad x_{ij} \leq y_i, \quad \sum_j x_{ij} = 1, \quad \sum_i y_i = p.$

The MIP:

$$\min \begin{bmatrix} 0 \\ 1400 \\ 1000 \\ 3500 \\ 0 \\ 5750 \\ 2000 \\ 4600 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^\top \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{12} \\ x_{22} \\ x_{32} \\ x_{13} \\ x_{23} \\ x_{33} \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

subject to $\mathbf{Az} = \mathbf{b}$ and $\mathbf{A}_{eq}\mathbf{z} = \mathbf{b}_{eq}$ and $\mathbf{l} \leq \mathbf{z}$ where $\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$ and

$$\mathbf{A} = \begin{bmatrix} & -\mathbf{I}_{3\times3} \\ \mathbf{I}_{9\times9} & -\mathbf{I}_{3\times3} \\ & -\mathbf{I}_{3\times3} \\ \mathbf{0}_{3\times9} & \mathbf{I}_{3\times3} \\ \mathbf{0}_{1\times9} & \mathbf{1}_{1\times3} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{0}_{9\times1} \\ \mathbf{1}_{3\times1} \\ p \end{bmatrix}, \quad \mathbf{A}_{eq} = \begin{bmatrix} \mathbf{1}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{1}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{1}_{1\times3} & \mathbf{0}_{1\times3} \end{bmatrix}, \quad \mathbf{b}_{eq} = \mathbf{1}_{3\times1}, \quad \mathbf{l} = \mathbf{0}_{12\times1}$$

Code: `z = intlinprog(c,set,A,b,Aeq,beq,l)`

Solutions :

- $p = 1$: $y_1 = x_{11} = x_{12} = x_{13} = 1$, else all $= 0$ — 10 pts
- $p = 2$: $y_2 = y_3 = x_{31} = x_{22} = x_{33} = 1$, else all $= 0$ — 10 pts

END.