Solution to CO327 (2021Spring) Assignment 5

Lecturer: Andersen Ang

August 2, 2021

1 Robust curve fitting (20 points)

Introduction This part refers to `Q2_main.m` and `Q2.mat` in the data file.

One of the task in data analysis is to find the pattern(s) in data. In this question you will perform a simple simple data analysis known as "Regression". More specifically, "Least Sqaures" and "Robust regression'.

About the problem You are given n points (x_i, y_i), $i = 1, 2, \ldots, n$ which you believe they are generated under the model $y = mx + c$. However, the data points are corrupted by noise and therefore the points are not exactly lying on a straight line.

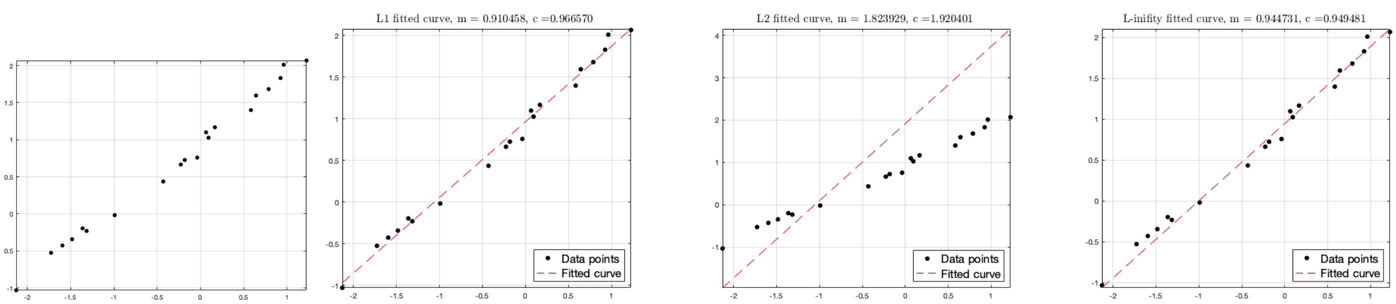


Figure 1: The input data and what you should get for different types of fitting.

For the data points, you propose the following model $y_i = mx_i + c + \epsilon_i, \quad i = 1, 2, \ldots, n$ where ϵ refers to the noise. Refer to the lecture on curve fitting, a way to find (m, c) is to minimize the norm $\mathbf{e} = [\epsilon_1, \epsilon_2, \ldots, \epsilon_n]$. Mathematically, we have the following so-called "linear model":

$$\mathbf{Ax} = \mathbf{b} + \mathbf{e}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} c \\ m \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}.$$

We can see that minimizing the norm of $\mathbf{e}$ is equivalent to minimizing $\|\mathbf{Ax} - \mathbf{b}\|_p$ for $p \in \{1, 2, \infty\}$. Your task: write 3 MATLAB codes that minimizes this function for $p = 1, 2, \infty$. Especially,

- For $p = 1$ or ∞, turn the problem as a LP and then solve it using `linprog`.

- For $p = 2$, consider minimizing $\|\mathbf{Ax} - \mathbf{b}\|_2^2$, turn the problem as a QP and then solve it using `quadprog`. Note that

$$\|\mathbf{Ax} - \mathbf{b}\|_2^2 = \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{b}^\top \mathbf{Ax} + \mathbf{b}^\top \mathbf{b}$$

After you obtain (m, c), plot the result. For details, see `Q2_main.m` in the data file.

Solution

ℓ_1 case

$$\min \; \hat{\mathbf{c}}^\top \hat{\mathbf{x}} \text{ s.t. } \hat{\mathbf{A}}\hat{\mathbf{x}} \leq \hat{\mathbf{b}}$$

where

$$\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \quad \hat{\mathbf{c}} = \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \mathbf{1}_{m\times 1} \end{bmatrix}, \quad \hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & -\mathbf{I}_m \\ -\mathbf{A} & -\mathbf{I}_m \end{bmatrix}, \quad \hat{\mathbf{b}} = \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}$$

ℓ_∞ case

$$\min \; \hat{\mathbf{c}}^\top \hat{\mathbf{x}} \text{ s.t. } \hat{\mathbf{A}}\hat{\mathbf{x}} \leq \hat{\mathbf{b}}$$

where

$$\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix}, \quad \hat{\mathbf{c}} = \begin{bmatrix} \mathbf{0}_{n\times 1} \\ 1 \end{bmatrix}, \quad \hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & -\mathbf{1}_{m\times 1} \\ -\mathbf{A} & -\mathbf{1}_{m\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix}, \quad \hat{\mathbf{b}} = \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \\ 0 \end{bmatrix}$$

or

$$\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix}, \quad \hat{\mathbf{c}} = \begin{bmatrix} \mathbf{0}_{n\times 1} \\ 1 \end{bmatrix}, \quad \hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & -\mathbf{1}_{m\times 1} \\ -\mathbf{A} & -\mathbf{1}_{m\times 1} \end{bmatrix}, \quad \hat{\mathbf{b}} = \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}$$

ℓ_2 case

Notice that quadprog solve

$$\min \mathbf{x}^\top \frac{\mathbf{H}}{2} \mathbf{x} + \mathbf{f}^\top \mathbf{x}$$

we have

$$\min \; \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{b}^\top \mathbf{Ax} \quad \Longrightarrow \quad \mathbf{H} = 2\mathbf{A}^\top \mathbf{A}, \;\; \mathbf{f} = -2\mathbf{b}^\top \mathbf{A}$$

or

$$\min \; \frac{1}{2}\mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - \mathbf{b}^\top \mathbf{Ax} \quad \Longrightarrow \quad \mathbf{H} = \mathbf{A}^\top \mathbf{A}, \;\; \mathbf{f} = -\mathbf{b}^\top \mathbf{A}$$

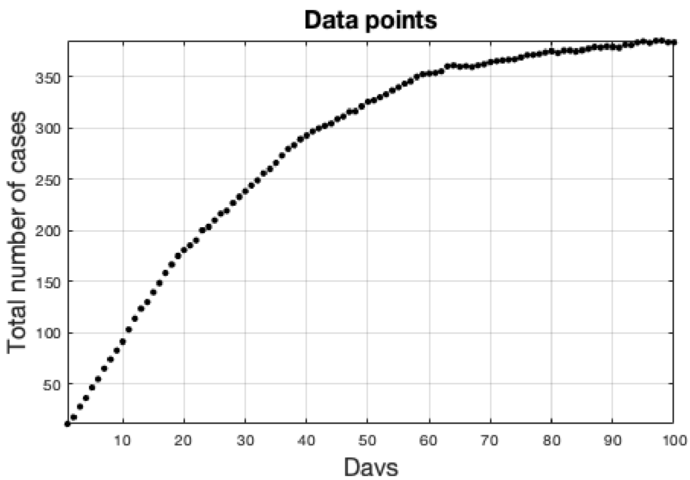
Marking

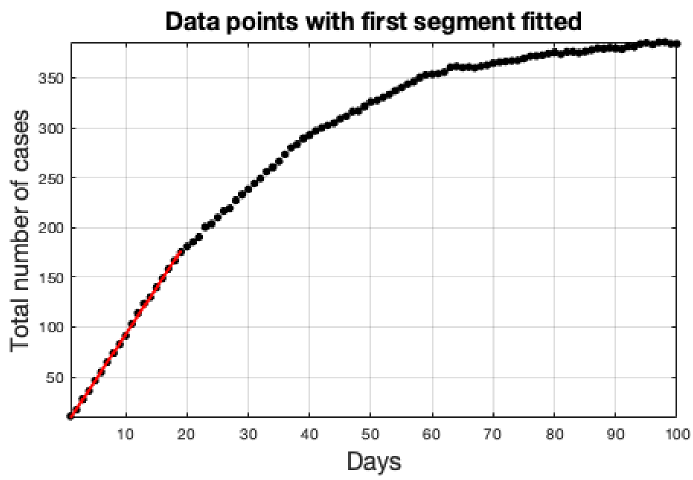
Correct code and correct solution for ℓ_1	6 pt.
Correct code and correct solution for ℓ_2	6 pt.
Correct code and correct solution for ℓ_∞	6 pt.
PDF	2 pt.

2 Change point detection (40 points)

This part refers to `Q3_main.m` in the data file.

You are given 100 days of record of total infected cases of an illness of a pandemic in a country. Mathematically, you are given 100 points (x_i, y_i), where x_i are integers from 1 to 100, representing the number of days since the outbreak of the pandemic, and y_i refer to the number of total infected cases.





The data is suspected to be generated under a piecewise-linear model as

$$\mathbf{y} = \min\{m_1\mathbf{x} + c_1, m_2\mathbf{x} + c_2, \ldots, m_r\mathbf{x} + c_r\}$$

with r mode (different m_j, c_j). Or in other words, you assume the data is generated by the model $y_i = m_j x_i + c_j$ with multiple sets of (m_j, c_j) that correspond to different period of time (different range of x_i). Your tasks are:

- Identify r, the number of mode in the data
- Identify these (m, c)
- Identify the change point (the day when the model changed).

What to do Write a function to perform a robust curve fitting on a set of points. then test on different segment of the data to identify each linear function. In other words, use the code you have developed in Q2. Here, instead of using all the points as in Q2, use a subset of points that you think they are belong to the same model.

Example The red line in the figure is obtained using the first 20 points in the data points. As the $\|\mathbf{A}_{\text{subset}}\mathbf{x}_{\text{subset}} - \mathbf{b}_{\text{subset}}\|_p$ is very low for these 20 points, hence we have high confidence that the first 20 points fall into the model of $m = 9.3238$ and $c = -0.67$. That is, we have $y = 9.3238x - 0.67$ for the period $1 \leq x_i \leq 20$.

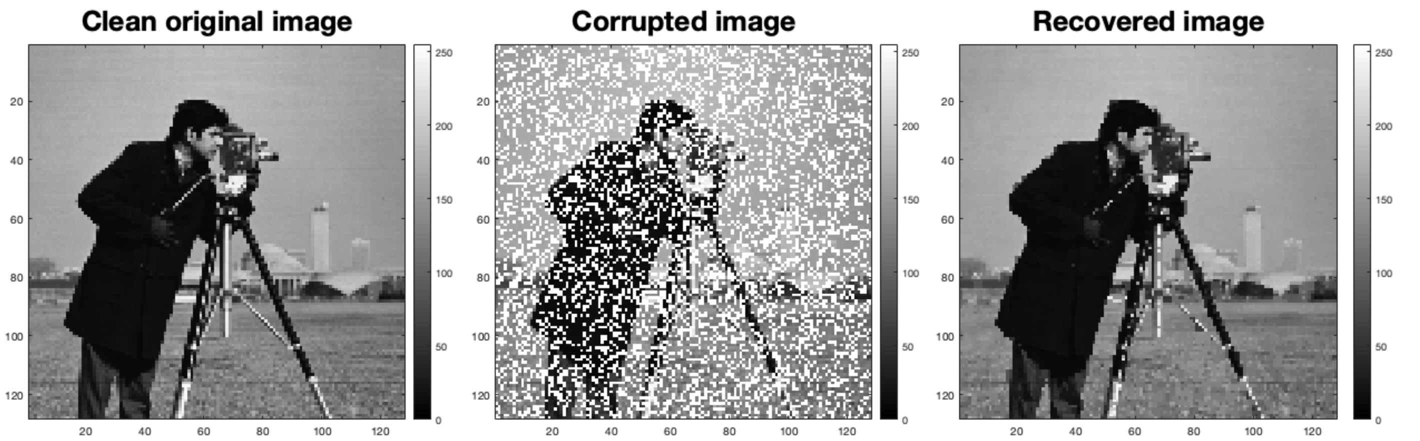
Sol There are 5 linear function with period 20:

Period 1	$1 \leq x_i \leq 20$	$m = 9.3238,\ c = -0.6575$
Period 2	$21 \leq x_i \leq 40$	$m = 5.6783,\ c = 67.8491$
Period 3	$41 \leq x_i \leq 60$	$m = 3.1260,\ c = 168.3103$
Period 4	$61 \leq x_i \leq 80$	$m = 1.0953,\ c = 287.7911$
Period 5	$81 \leq x_i \leq 100$	$m = 0.5900,\ c = 327.2611$

Marking Code (easy to read? tidy? no bug?)	10 pt.
Solution	15 pt., 3 each
Logic / model (correct model, correct approach, correct idea)	10 pt.
PDF (explanation, description, discussion)	5 pt.

3 Image completion (80 points)

In this question you are going to perform a "dark magic" in image processing called image inpainting (or mathematically called image completion). The task is simple: you are given a "broken image", the goal is to repair the image. Here an image (a n-by-n matrix or a n^2 vectorized vector) contains numbers from 0 to 254, and 255 represents a broken pixel value.



Refer to the lecture on image inpainting, the mathematics of such "magic" is the following optimization problem

$$\min \ \|\mathbf{Ex}\|_1 \quad \text{s.t.} \quad \mathbf{Sx} = \hat{\mathbf{x}}_\Omega, \ \mathbf{x} \geq \mathbf{0}, \tag{*}$$

where $\mathbf{x} \in \mathbb{R}^{n^2}$ is the vectorized target image, $\|\mathbf{Ex}\|_1$ is called the "Total Variation" (TV) of the vector $\mathbf{x}$, the matrix $\mathbf{S} \in \mathbb{R}_+^{|\Omega| \times n^2}$ is a sub-matrix of $\mathbf{I}_{n^2}$ consists of rows labeled in the set Ω, and $\hat{\mathbf{x}}_\Omega \in \mathbb{R}_+^{|\Omega|}$ is the clean part of the observed image, with $|\Omega| < n^2$ number of entries.

LP Given $\mathbf{E} \in \mathbb{R}^{2n(n-1) \times n^2}, \mathbf{S} \in \mathbb{R}_+^{|\Omega| \times n^2}$, and $\hat{\mathbf{x}}_\Omega \in \mathbb{R}_+^{|\Omega|}$, write a program to solve Problem (*). Plot the recovered image. See `Q4_main.m` in the data file for details.

Relaxations to QP Now instead of solving Problem (*), we consider solving the following problem:

$$\min \|\mathbf{Ex}\|_1 + \frac{1}{2}\|\mathbf{Sx} - \hat{\mathbf{x}}_\Omega\|_2^2 \ \text{ s.t. } \ \mathbf{x} \geq \mathbf{0} \tag{**}$$

Write a program to solve Problem (**), plot the recovered image, and compare the recovered image to the one from solving Problem (*). Hint: Make use of `quadprog(Q,p,A,b,Aeq,beq,l)`.

General formulation Solve the following

$$\min \|\mathbf{Ex}\|_p + \|\mathbf{Sx} - \hat{\mathbf{x}}_\Omega\|_q \ \text{ s.t. } \ \mathbf{x} \geq \mathbf{0} \tag{***}$$

for $p, q \in \{1, \infty\}$. Write a series of programs to solve Problem (***), plot these recovered images, compare the recovered image to the one from solving Problem (*) and (**).

General hint If you implemented everything correct, the recovered image should looks like the original clean image. There are 2 images in the data file, try with `mario.m` (the smaller one) first.

Solution. The important trick here: see midterm Q3.2 !

$$\min_{\mathbf{x}} \|\mathbf{Ax}\|_1 \iff \min_{\mathbf{x},\mathbf{u}} \mathbf{1}^\top \mathbf{u} \ \text{ s.t. } \ -\mathbf{u} \leq \mathbf{Ax} \leq \mathbf{u} \qquad \text{(Important trick)}$$

(*)

Let the size of $\mathbf{E}$ be $M \times N$, then

$$\min \ \|\mathbf{E}\mathbf{x}\|_1 \ \text{ s.t. } \ \mathbf{S}\mathbf{x} = \hat{\mathbf{x}}_\Omega,\ \mathbf{x} \geq \mathbf{0}, \quad \Longleftrightarrow \quad \min \hat{\mathbf{c}}^\top \hat{\mathbf{x}} \ \text{ s.t. } \ \hat{\mathbf{A}}\hat{\mathbf{x}} \leq \hat{\mathbf{b}},\ \hat{\mathbf{A}}_{\text{eq}}\hat{\mathbf{x}} = \hat{\mathbf{b}}_{\text{eq}},\ \hat{\mathbf{x}} \geq \mathbf{0}$$

where $\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \in \mathbb{R}^{(n^2+M)\times 1}$, $\hat{\mathbf{c}} = \begin{bmatrix} \mathbf{0}_{n^2\times 1} \\ \mathbf{1}_{M\times 1} \end{bmatrix} \in \mathbb{R}^{(n^2+M)\times 1}$, and

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{E} & -\mathbf{I}_M \\ -\mathbf{E} & -\mathbf{I}_M \end{bmatrix},\ \hat{\mathbf{b}} = \mathbf{0}_{2M\times 1},\ \hat{\mathbf{A}}_{\text{eq}} = \begin{bmatrix} \mathbf{S} & \mathbf{0}_{|\Omega|\times M} \end{bmatrix},\ \hat{\mathbf{b}}_{\text{eq}} = \hat{\mathbf{x}}_\Omega$$

(**)

$$\min_{\mathbf{x}\geq\mathbf{0}} \|\mathbf{E}\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{S}\mathbf{x} - \hat{\mathbf{x}}_\Omega\|_2^2 \quad \Longleftrightarrow \quad \min_{\hat{\mathbf{x}}\geq\mathbf{0}} \frac{1}{2}\hat{\mathbf{x}}^\top \mathbf{Q}\hat{\mathbf{x}} + \mathbf{p}^\top \hat{\mathbf{x}} \ \text{ s.t. } \ \hat{\mathbf{A}}\hat{\mathbf{x}} \leq \hat{\mathbf{b}}$$

where where $\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \in \mathbb{R}^{(n^2+M)\times 1}$ and

$$\mathbf{Q} = \begin{bmatrix} \mathbf{S} & \mathbf{0}_{|\Omega|\times M} \\ \mathbf{0}_{M\times n^2} & \mathbf{0}_{M\times M} \end{bmatrix},\ \mathbf{p} = \begin{bmatrix} -\mathbf{S}^\top \hat{\mathbf{x}}_\Omega \\ \mathbf{1}_M \end{bmatrix},\ \hat{\mathbf{A}} = \begin{bmatrix} \mathbf{E} & -\mathbf{I}_M \\ -\mathbf{E} & -\mathbf{I}_M \end{bmatrix},\ \hat{\mathbf{b}} = \mathbf{0}_{2M\times 1},$$

(p, q)

$$\min \begin{bmatrix} \mathbf{0}_{n^2} \\ \mathbf{1}_{M\times 1} \\ \mathbf{1}_{|\Omega|\times 1} \end{bmatrix}^\top \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} \ \text{ s.t. } \ \begin{bmatrix} -\mathbf{S} & \mathbf{0}_{|\Omega|\times M} & -\mathbf{I}_{|\Omega|} \\ \mathbf{S} & \mathbf{0}_{|\Omega|\times M} & -\mathbf{I}_{|\Omega|} \\ -\mathbf{E} & -\mathbf{I}_M & \mathbf{0}_{M\times|\Omega|} \\ \mathbf{E} & -\mathbf{I}_M & \mathbf{0}_{M\times|\Omega|} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} \leq \begin{bmatrix} -\hat{\mathbf{x}}_\Omega \\ \hat{\mathbf{x}}_\Omega \\ \mathbf{0}_{|\Omega|\times 1} \\ \mathbf{0}_{|\Omega|\times 1} \end{bmatrix} \qquad (1,1)$$

$$\min \begin{bmatrix} \mathbf{0}_{n^2} \\ \mathbf{1}_{M\times 1} \\ 1 \end{bmatrix}^\top \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ t \end{bmatrix} \ \text{ s.t. } \ \begin{bmatrix} -\mathbf{S} & \mathbf{0}_{|\Omega|\times M} & -\mathbf{1}_{|\Omega|\times 1} \\ \mathbf{S} & \mathbf{0}_{|\Omega|\times M} & -\mathbf{1}_{|\Omega|\times 1} \\ -\mathbf{E} & -\mathbf{I}_M & \mathbf{0}_{M\times 1} \\ \mathbf{E} & -\mathbf{I}_M & \mathbf{0}_{M\times 1} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ t \end{bmatrix} \leq \begin{bmatrix} -\hat{\mathbf{x}}_\Omega \\ \hat{\mathbf{x}}_\Omega \\ \mathbf{0}_{|\Omega|\times 1} \\ \mathbf{0}_{|\Omega|\times 1} \end{bmatrix} \qquad (1,\infty)$$

$$\min \begin{bmatrix} \mathbf{0}_{n^2} \\ 1 \\ \mathbf{1}_{|\Omega|\times 1} \end{bmatrix}^\top \begin{bmatrix} \mathbf{x} \\ s \\ \mathbf{z} \end{bmatrix} \ \text{ s.t. } \ \begin{bmatrix} -\mathbf{S} & \mathbf{0}_{|\Omega|\times 1} & -\mathbf{I}_{|\Omega|} \\ \mathbf{S} & \mathbf{0}_{|\Omega|\times 1} & -\mathbf{I}_{|\Omega|} \\ -\mathbf{E} & -\mathbf{1}_{M\times 1} & \mathbf{0}_{M\times|\Omega|} \\ \mathbf{E} & -\mathbf{1}_{M\times 1} & \mathbf{0}_{M\times|\Omega|} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ s \\ \mathbf{z} \end{bmatrix} \leq \begin{bmatrix} -\hat{\mathbf{x}}_\Omega \\ \hat{\mathbf{x}}_\Omega \\ \mathbf{0}_{|\Omega|\times 1} \\ \mathbf{0}_{|\Omega|\times 1} \end{bmatrix} \qquad (\infty,1)$$

$$\min \begin{bmatrix} \mathbf{0}_{n^2} \\ 1 \\ 1 \end{bmatrix}^\top \begin{bmatrix} \mathbf{x} \\ s \\ t \end{bmatrix} \ \text{ s.t. } \ \begin{bmatrix} -\mathbf{S} & \mathbf{0}_{|\Omega|\times 1} & -\mathbf{1}_{|\Omega|\times 1} \\ \mathbf{S} & \mathbf{0}_{|\Omega|\times 1} & -\mathbf{1}_{|\Omega|\times 1} \\ -\mathbf{E} & -\mathbf{1}_{M\times 1} & \mathbf{0}_{M\times 1} \\ \mathbf{E} & -\mathbf{1}_{M\times 1} & \mathbf{0}_{M\times 1} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ s \\ t \end{bmatrix} \leq \begin{bmatrix} -\hat{\mathbf{x}}_\Omega \\ \hat{\mathbf{x}}_\Omega \\ \mathbf{0}_{|\Omega|\times 1} \\ \mathbf{0}_{|\Omega|\times 1} \end{bmatrix} \qquad (\infty,\infty)$$

where $\mathbf{I}_a$ is identity matrix of $a \times a$ and $\mathbf{I}_{a\times b}$ is identity matrix of $a \times b$ by chopping rows or adding zero rows based on a, b.

Marking (*), code, solution, model formulation, PDF explanation 20 pt.
(**), code, solution, model formulation, PDF explanation 20 pt.
(***), code, solution, model formulation, PDF explanation 40 pt.

END of assignment.