CO327 Deterministic OR Models (2021-Spring)

Airport traffic control, max-min problem and the extreme-case modeling trick

Instructor: Andersen Ang

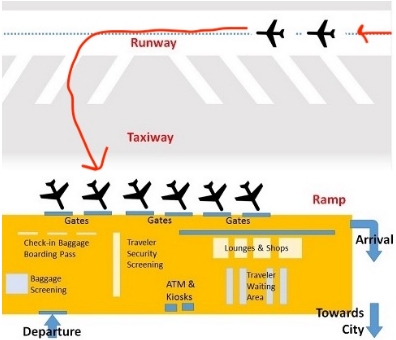
Combinatorics and Optimization, U.Waterloo, Canada

msxang@uwaterloo.ca, where $\mathbf{x} = \lfloor \pi \rfloor$

Homepage: angms.science

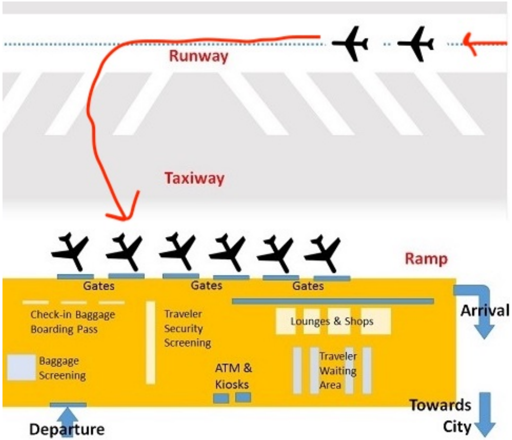
First draft: April 28, 2021 Last update: May 26, 2021

Airport traffic control



- Plane A landing on runway
- Plane A go to the taxiway
- Plane A go to gates and drop people
- Repeat on plane B
- What's the task: determine the actual arrival time for each plane to land

Safety issue



- Air planes don't want to see each other.
- Their landing time should be different.
- What is safe: the time difference between their landing time is as large as possible

Modeling the problem

- Let there be $1, 2, \dots, n$ plane.
- Let t_i be the actual landing time of plane i.
- Then, the difference in time between plane i and plane j is

$$t_i - t_j$$

- This is NOT a good modeling because you can take any $j \neq i$
  - you may take into account the plane j that has already landed 999 years before plane i
  - you may take into account the plane j that has will arrive 999 years after plane i
  - What really matters should be the plane j right after plane i

Modeling the problem

- Let there be $1, 2, \ldots, n$ plane *such that their arrival time are arranged in accending order*

$$0 \leq t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_n$$

- Let t_i be the actual landing time of plane i.
- Then, the difference in time between plane i and plane $i+1$ is

$$t_{i+1} - t_i$$

- This quantity $t_{i+1} - t_i$ is always nonnegative

Worst-case modeling

- Worst-case: the $(i, i+1)$ pair that their time difference is the smallest

$$\min_i \left\{ \cdots , t_i - t_{i-1} , t_{i+1} - t_i , t_{i+2} - t_{i+1} , \cdots \right\}.$$

- Notation: denote

$$\min_i (t_{i+1} - t_i) = \min_i \left\{ \cdots , t_i - t_{i-1}, t_{i+1} - t_i, t_{i+2} - t_{i+1}, \cdots \right\}.$$

- What is your goal: maximize such time difference

$$\max \left(\min_i (t_{i+1} - t_i) \right)$$

- A max-min problem !

- A composite optimization problem !

Finishing the modeling: constraint

- t_i cannot be chosen arbitrarily: it has to be within a time range

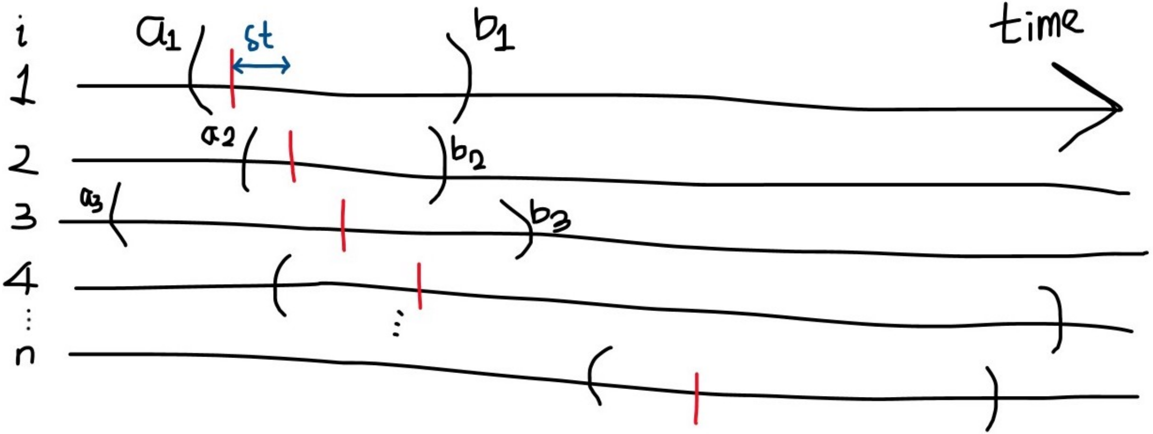
$$a_i \leq t_i \leq b_i$$

- So the airport traffic control problem is

$$\begin{aligned} \max \quad & \left(\min_i \, (t_{i+1} - t_i) \right) \\ \text{s.t.} \quad & a_i \leq t_i \leq b_i \ \forall i \end{aligned}$$

- Question: is this a linear programming problem?

Visualizing the airport traffic control problem:
As a 1-dimensional match placement problem



"Place the red bars such that the smallest gap is maximized"

The worst-case trick

- The problem

$$\begin{aligned} \max \quad & \left(\min_i \, (t_{i+1} - t_i) \right) \\ \text{s.t.} \quad & a_i \leq t_i \leq b_i \; \forall i \end{aligned}$$

  has max and min.

- Modeling technique: introduce a new variable.
  Introducing the lower bound

$$\Delta := \min_i \, (t_{i+1} - t_i).$$

- Because Δ is the lower bound, so all the time difference must be above Δ

$$t_{i+1} - t_i \geq \Delta$$

  or

$$t_{i+1} - t_i - \Delta \geq 0.$$

Airport traffic control problem in LP form

$$
\begin{aligned}
\max \quad & \Delta \\
\text{s.t.} \quad & a_i \le t_i \le b_i \ \forall i \\
& t_{i+1} - t_i - \Delta \ge 0 \ \forall i
\end{aligned}
$$

- Not just for airport traffic control, this model also works for train traffic control, ship traffic control, bus traffic control
- Field: discrete event scheduling

A concrete example

There are 3 planes:

$$
\begin{aligned}
\max \quad & \left(\min_i \; \{t_2 - t_1, t_3 - t_2\} \right) \\
\text{s.t.} \quad & 0 \le t_1 \le 2 \\
& 1 \le t_2 \le 3 \\
& 2 \le t_3 \le 4
\end{aligned}
$$

In LP form

$$
\begin{aligned}
\max \quad & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^\top \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \Delta \end{bmatrix} \\
\text{s.t.} \quad & 0 \le t_1 \le 2 \\
& 1 \le t_2 \le 3 \\
& 2 \le t_3 \le 4 \\
& t_2 - t_1 - \Delta \ge 0 \\
& t_3 - t_2 - \Delta \ge 0
\end{aligned}
$$

Assignment: write this problem in standard form and canonical form.

The extreme-case modeling trick

- The problem

$$\begin{array}{rl} \max & \{x_1, x_2, x_3, \cdots\} \\ \text{s.t.} & \text{some constraints} \end{array}$$

  is equivalent to

$$\begin{array}{rl} \max & y \\ \text{s.t.} & \text{some constraints} \\ & x_1 \leq y \\ & x_2 \leq y \\ & \vdots \end{array}$$

  The trick: introduce a upper bound y.

- Similar trick for min.
- This trick is useful
  - It "reduces" the min{} or max{} (with multiple arguments inside {}) to min or max of just one variable.
  - It simplifies composite max-min (min-max) problems.

Summary

- Airport traffic control
- Max-min problem
- Visualization of “1-dimensional match placement”
- The extreme-case mathematical trick

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