Solution to CO327 (2022Spring) Assignment 1

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1 Solving system of linear inequalities (8 points)

Determine if the following system of inequalities has a solution.

- If the system has a solution, give one solution.
- If the system has no solution, explain why.

Show your working steps.

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 2x_3 & \geq & 1 \\ -x_1 & + & x_2 & + & x_3 & \geq & 2 \\ x_1 & - & x_2 & + & x_3 & \geq & 1 \\ & - & x_2 & - & 3x_3 & \geq & 0 \end{array}$$

1.1 Solution

The system has no solution. (2 point)

First label the inequalities.

$$\begin{array}{rcrcrcll} x_1 & + & 2x_2 & + & 2x_3 & \geq & 1 & (1) \\ -x_1 & + & x_2 & + & x_3 & \geq & 2 & (2) \\ x_1 & - & x_2 & + & x_3 & \geq & 1 & (3) \\ & - & x_2 & - & 3x_3 & \geq & 0 & (4) \end{array} \qquad \text{(2 points)}$$

The system is equivalent to

$$\begin{array}{rcrcrcll} & & 3x_2 & + & 3x_3 & \geq & 3 & (1)+(2) \\ & & & & 2x_3 & \geq & 3 & (2)+(3) \\ & - & x_2 & - & 3x_3 & \geq & 0 & (4) \end{array} \qquad \text{(2 points)}$$

Let $(5) = \frac{(1)+(2)}{3}$

$$\begin{array}{rcrcrcll} & & x_2 & + & x_3 & \geq & 1 & (5) \\ & - & x_2 & - & x_3 & \geq & 3 & (2)+(3)+(4) \end{array} \qquad \text{(1 point)}$$

We get contraction if we add (5) to (2)+(3)+(4)

$$0 \geq 4. \qquad \text{(1 point)}$$

So there is no solution.

Marking: No solution 2 points, showing no solution by contradiction 6 points.

2 Transform to canonical form and standard form (22 points)

2.1 Transform the LP into canonical form, shows the working steps (8 points).

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & 0 \\ \text{s.t.} \quad & 1x_1 + 3x_2 = 5 \\ & 2x_1 + 4x_3 \geq 6 \\ & 2 \leq x_2 \leq 4 \end{aligned}$$

Also write down the vector $\mathbf{c}, \mathbf{b}$ and matrix $\mathbf{A}$.

min 0 becomes max 0 (1 point).

$1x_1 + 3x_2 = 5$ becomes $1x_1 + 3x_2 \leq 5$ and $1x_1 + 3x_2 \geq 5 \implies -1x_1 - 3x_2 \leq -5$ (1 point)

$2x_1 + 4x_3 \geq 6$ becomes $-2x_1 - 4x_3 \leq -6$ (1 point)

$2 \leq x_2 \leq 4$ becomes $-1x_2 \leq -2$ and $x_2 \leq 4$ (1 point)

So now

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & [0\ 0\ 0]\mathbf{x} \\ \text{s.t.} \quad & x_1 + 3x_2 \leq 5 \\ & -1x_1 - 3x_2 \leq -5 \\ & -2x_1 - 4x_3 \leq -6 \\ & -1x_2 \leq -2 \\ & x_2 \leq 4 \end{aligned} \quad \text{or} \quad \begin{aligned} \max_{x_1,x_2,x_3} \quad & [0\ 0\ 0]\mathbf{x} \\ \text{s.t.} \quad & x_1 + 3x_2 \leq 5 \\ & -1x_1 - 3x_2 \leq -5 \\ & -1x_1 - 2x_3 \leq -3 \\ & -1x_2 \leq -2 \\ & x_2 \leq 4 \end{aligned} \qquad \text{(1 point)}$$

and $\mathbf{c} = [0, 0, 0]^\top$ (1 point), and

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \\ -2 & 0 & -4 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 5 \\ -5 \\ -6 \\ -2 \\ 4 \end{bmatrix} \quad \text{or} \quad \mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \\ -1 & 0 & -2 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 5 \\ -5 \\ -3 \\ -2 \\ 4 \end{bmatrix} \qquad \text{(2 points)}$$

2.2 Transform the LP into standard form (7 points)

$$\begin{aligned} \min_{x_1,x_2} \quad & 5x_1 + 2x_2 \\ \text{s.t.} \quad & 6x_1 + x_2 \geq 6 \\ & 4x_1 + 3x_2 \leq 12 \end{aligned}$$

Let $x_1 = x_1^+ - x_1^-$, $x_1^+ \geq 0, x_1^- \geq 0$ and $x_2 = x_2^+ - x_2^-$, $x_2^+ \geq 0, x_2^- \geq 0$, and introduce two slack variables $s_1 \geq 0, s_2 \geq 0$ and the LP becomes

$$\begin{aligned} \max_{x_1^+,x_1^-,x_2^+,x_2^-,s_1,s_2} \quad & -5x_1^+ + 5x_1^- - 2x_2^+ + 2x_2^- \\ \text{s.t.} \quad & 6x_1^+ - 6x_1^- + x_2^+ - x_2^- - s_1 = 6 \\ & 4x_1^+ - 4x_1^- + 3x_2^+ - 3x_2^- + s_2 = 12 \\ & x_1^+, x_1^-, x_2^+, x_2^-, s_1, s_2 \geq 0 \end{aligned}$$

1 point for $x_1 = x_1^+ - x_1^-$, $x_1^+ \geq 0, x_1^- \geq 0$ and $x_2 = x_2^+ - x_2^-$, $x_2^+ \geq 0, x_2^- \geq 0$.

1 point for $s_1 \geq 0, s_2 \geq 0$

1 point for max

1 point for correct objective function.

2 points for equality constraints (1 point each)

1 point for all the inequality constraints

2.3 Transform the LP of 2.1 to standard form (7 points).

The LP of 2.1

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & 0 \\ \text{s.t.} \quad & 1x_1 + 3x_2 = 5 \\ & 2x_1 + 4x_3 \geq 6 \\ & 2 \leq x_2 \leq 4 \end{aligned} \quad \rightarrow \quad \begin{aligned} \max_{x_1,x_2,x_3} \quad & 0 \\ \text{s.t.} \quad & 1x_1^+ - x_1^- + 3(2+s_1) = 5 \\ & 2x_1^+ - 2x_1^- + 4x_3^+ - 4x_3^- - s_2 = 6 \\ & s_1 \leq 2 \\ & x_1^+, x_1^-, x_3^+, x_3^-, s_1, s_2 \geq 0 \end{aligned}$$

where 1. $x_1 = x_1^+ - x_1^-$, $x_1^+ \geq 0, x_1^-, \geq 0$.

2. $x_2 = 2 + s_1$, $s_1 \geq 0$

3. $x_3 = x_3^+ - x_3^-$, $x_3^+ \geq 0, x_3^-, \geq 0$.

4. Slack variable s_2 for inequality constraint.

5. min to max

Then, 6. Add slack variable s_3 for $s_1 \leq 2$

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & 0 \\ \text{s.t.} \quad & 1x_1^+ - x_1^- + 3s_1 = -1 \\ & 2x_1^+ - 2x_1^- + 4x_3^+ - 4x_3^- - s_2 = 6 \\ & s_1 + s_3 = 2 \\ & x_1^+, x_1^-, x_3^+, x_3^-, s_1, s_2, s_3 \geq 0 \end{aligned}$$

1 point for max

3 points for 3 correct equality constraints (1 point each)

1 point for inequity constraints

1 point for appearance of all $x_1^+, x_1^-, x_3^+, x_3^-$

1 point for appearance of all $s_1 s_2, s_3$

END of assignment 1.