Solution to CO327 (2022Spring) Assignment 3

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1 Simple problems (9 points)

Find the optimal $\mathbf{x}$. If the problem has no solution, explain why.

$$(1)\quad \begin{array}{ll} \max\limits_{x_1,x_2} & 7x_1 + 5x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 100 \\ & 4x_1 + 3x_2 \leq 240 \\ & x_1 \geq 0, x_2 \geq 0 \end{array} \quad (2)\quad \begin{array}{ll} \min\limits_{x_1,x_2,x_3} & -5x_1 - 4x_2 - 6x_3 \\ \text{s.t.} & x_1 - x_2 + x_3 \leq 20 \\ & -3x_1 - 2x_2 - 4x_3 \geq -42 \\ & 3x_1 + 2x_2 \leq 30 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{array} \quad (3)\quad \begin{array}{ll} \max\limits_{x_1,x_2} & 150x_1 + 175x_2 \\ \text{s.t.} & 7x_1 + 11x_2 \leq 77 \\ & 10x_1 + 8x_2 \leq 80 \\ & 1 \leq x_1 \leq 9 \\ & 3 \leq x_2 \leq 6 \end{array}$$

Sol

1. $\mathbf{x}^* = [30, 40]^\top$.
2. $\mathbf{x}^* = [0, 15, 3]^\top$.
3. $\mathbf{x}^* = [4.8889, 3.8889]^\top$

Marking:

- Correct solution: 3 points and no need to check the code (suppose it is correct).
- Incorrect solution
  - 2 points if the code has minor error.
  - 1 point if the code has major error.
  - 0 point if the code is all wrong.

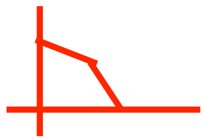
2 LP relaxation of IP (10 points)

1. Draw the feasible region of the following constraints: $x_1 + x_2 \leq 6$, $5x_1 + 9x_2 \leq 45$, $x_1 \geq 0, x_2 \geq 0$.
2. Find $\mathbf{x}_{LP} = [x_1, x_2]^\top$ that maximizes $5x_1 + 8x_2$ subject to $x_1 + x_2 \leq 6$, $5x_1 + 9x_2 \leq 45$, $x_1 \geq 0, x_2 \geq 0$. If the problem has no solution, explain why.
3. Find $\mathbf{x}_{IP} = [x_1, x_2]^\top$ that maximizes $5x_1 + 8x_2$ subject to $x_1 + x_2 \leq 6$, $5x_1 + 9x_2 \leq 45$, $x_1 \geq 0, x_2 \geq 0$ and x_1, x_2 are integers. If the problem has no solution, explain why.
4. (On LP relaxation) Round the solution $\mathbf{x}$ for the LP problem to:

- The closest integer $\mathbf{x}_{LP}^{Round}$
- The closest integer that is feasible for the constraint $\mathbf{x}_{LP}^{Round,feasible}$

Let $f(\mathbf{x})$ be the objective function values at $\mathbf{x}$. Compare the following values: $f(\mathbf{x}_{LP})$, $f(\mathbf{x}_{IP})$, $f(\mathbf{x}_{LP}^{Round})$ and $f(\mathbf{x}_{LP}^{Round,feasible})$. Explain why these objective function values are different. What can you tell about the effectiveness of LP relaxation on solving this IP?

Sol

1. Figure 1 point 
2. $\mathbf{x}_{LP}^* = [2.25, 3.75]$ (1 pt), optimal val = -41.25 for the matlab min problem, and = +41.25 for the original problem (1 pt) (note that MATLAB uses min but the problem uses max, so there is a sign difference)
3. $\mathbf{x}_{IP}^* = [0, 5]$ (1 pt), optimal val = -40 (matlab min problem), and = +40 for the original problem (1 pt)
4. Rounded $\mathbf{x}_{LP}^* = [2, 4]$ (1 pt), objective value = +42, but such point is infeasible. Such objective value 42 is higher than 40 for the IP because the point is infeasible (i.e., a constraint is relaxed). (1 pt)

   Rounded feasible $\mathbf{x}_{LP}^* = [2, 3]$ or $[3, 3]$ (1 pt), objective value = +34 (or +39), but such feasible point is far from the optimal solution $[0, 5]$. (1 pt)

   LP relaxation does not work here as rounding will round $\mathbf{x}_{LP}$ far away from $\mathbf{x}_{IP}$. (1 pt)

   (* give points as long as the arguments make sense)

3 BIP (6 points)

1. Find $\mathbf{x} = [x_1, x_2, x_3, x_4]^\top$ that maximizes $8x_1+11x_2-6x_3+4x_4$ subject to $5x_1+7x_2-4x_3+3x_4 \leq 14$ and all $x_i \in \{0, 1\}$. If the problem has no solution, explain why.
2. Convert the previous problem into a Knapsack problem and solve the new BIP. Does the new problem share the same solution to the previous one? If no, explain why.

Sol

1. Reformulate the problem to

   $$\min \quad -[8, 11, -6, 4]\mathbf{x} \quad \text{s.t.} \quad [5, 7, -4, 3]\mathbf{x} \leq 14, x \in \{0, 1\}$$

   MATLAB code `intlinprog(c,[1:4],A,b,[],[],zeros(4,1),ones(4,1))` gives $\mathbf{x}^* = [1, 1, 0, 0]^\top$ (2 pts), optimal cost = -19 for the MATLAB min problem and = +19 for the original problem (2 pts).
2. Knapsack problem : maximize $8x_1 + 11x_2 + 6y_3 + 4x_4$**-6** subject to $5x_1 + 7x_2 + 4y_3 + 3x_4 \leq 18$, all $x, y \in \{0, 1\}$
   - Not the same sol, because they give different solution $\mathbf{x}$ (or similar explanation) only get 1pt
   - Same sol get all 2 pts.

   (If the student does not show the derivation (see below), -1 point)

   (If the student shows the derivation but wrong, -1 point)

Detail explanation why same solution Students may think that the problems give different **x** so they have different solution, this is true. However, both problem share the same optimal value. For $\mathbf{x} = [1, 1, 1, 0]$ in part 2 it gives $8(1) + 11(1) + 6(1) + 4(0)$ **-6** $= 19$. The trap here is the constant term -6.

END.

The derivation for part 2

original problem

$$\min_x -\begin{bmatrix} 8 \\ 11 \\ -6 \\ 4 \end{bmatrix}^T x \quad \text{s.t.} \begin{bmatrix} 5 \\ 7 \\ -4 \\ 3 \end{bmatrix}^T x \leq 14, \quad x_i \in \{0,1\}$$

$$\Downarrow$$

$$\max_x \begin{bmatrix} 8 \\ 11 \\ -6 \\ 4 \end{bmatrix}^T x \quad \text{s.t.} \begin{bmatrix} 5 \\ 7 \\ -4 \\ 3 \end{bmatrix}^T x \leq 14, \quad x_i \in \{0,1\}$$

In Knapsack we want all x has positive coefficient in the $c^T x$

so $-6x_3 \Longrightarrow -6(1-y_3)$, i.e. introduce $x_3 = 1 - y_3$

Now we have

$$\max_{x,y} \begin{bmatrix} 8 \\ 11 \\ +6 \\ 4 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ y_3 \\ x_4 \end{bmatrix} - 6 \quad \text{s.t.} \begin{bmatrix} 5 \\ 7 \\ +4 \\ 3 \end{bmatrix}^T \begin{pmatrix} x_1 \\ x_2 \\ y_3 \\ x_4 \end{pmatrix} \leq 18$$

-6: constant term

$+6$: positive!

$+4$: positive!

18: changed!