Solution to CO327 (2022Spring) Assignment 4

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1 Resource allocation (20 points)

1.1 Voting visit (7 points)

You are the next US presidential candidate. You must decide which states to visit in the 10 days before the election. Your goal is to increase the number of votes by the largest possible amount. Your election team provide you the following data

State	Vote ($\times 10^3$) increase by visit	Days required for visit
1	10	4
2	20	3
3	40	3
4	90	4
5	30	3
6	10	1

Formulate this problem. State clearly your decision variable(s), objective function and constraint(s). Solve this problem: which states should be visited? How many votes will be generated by these visits?

Solution Decision variable: x_i = decision to visit state i or not

$$x_1, \dots, x_6 \qquad \text{(1 point, 1/6 each)}$$

Objective function: sum of vote increase

$$10x_1 + 20x_2 + 40x_3 + 90x_4 + 30x_5 + 10x_6 \qquad \text{(1 point, either 1 or 0)}$$

Explicit constraint 1: 10 days limit

$$4x_1 + 3x_2 + 3x_3 + 4x_4 + 3x_5 + x_6 \leq 10 \qquad \text{(1 point)}$$

Implicit constraint 1: binary constraint

$$x_i \in \{0, 1\} \quad \forall i. \qquad \text{(1 point, 1/6 each)}$$

So the BIP is

$$\begin{aligned} \max_{x_{11},\dots,x_{33}} \quad & 10x_1 + 20x_2 + 40x_3 + 90x_4 + 30x_5 + 10x_6 \\ \text{s.t.} \quad & 4x_1 + 3x_2 + 3x_3 + 4x_4 + 3x_5 + x_6 \leq 10 \\ & x_i \in \{0, 1\} \quad \forall i. \end{aligned} \qquad \text{(1 point for the max)}$$

(5 points if the student directly write down the BIP and it is all correct. Otherwise, deduct the points according to the missing portion.)

Solution: visit states 3,4,5. (1 point) Vote increase: 40+90+30 = 160k. (1 point)

1.2 Voting visit and ads (13 points)

Now you are down to the last 5 days of the campaign. You have $300000 left and three key states appear likely to swing the election one way or other. Each state can be visited, or a TV ad series can be purchased. Your election team provide you the following data

State	Action	Vote ($\times 10^3$) increase by visit	Days required for visit	Cost ($\times 10^3$)
1	Visit	100	4	200
1	Ads	50	0	100
2	Visit	80	4	150
2	Ads	40	0	90
3	Visit	20	1	45
3	Ads	15	0	30

Now assume

- Case a: visit and ads on the same state are not mutually exclusive (6 points)
- Case b: visit and ads on the same state are mutually exclusive (7 points)

Formulate these problems. State clearly your decision variable(s), objective function and constraint(s).
* You do not need to solve the program.

Solution Case a Decision variable: x_{ij} = decision on state i by action j

$$x_{11}, x_{12}, x_{21}, \ldots, x_{32} \qquad \text{(1 point, 1/6 each)}$$

Objective function: sum of vote increase

$$100x_{11} + 50x_{12} + 80x_{21} + 40x_{22} + 20x_{31} + 15x_{32} \qquad \text{(1 point, either 1 or 0)}$$

Explicit constraint 1: 5 days limit

$$4x_{11} + 0x_{12} + 4x_{21} + 0x_{22} + x_{31} + 0x_{32} \leq 5 \qquad \text{(1 point)}$$

Note: those zero term can be omitted.
Explicit constraint 2: 300000 (=300k) budget limit

$$200x_{11} + 100x_{12} + 150x_{21} + 90x_{22} + 45x_{31} + 30x_{32} \leq 300 \qquad \text{(1 point)}$$

Implicit constraint 1: binary constraint

$$x_{ij} \in \{0, 1\} \quad \forall i, j \qquad \text{(1 point, 1/6 each)}$$

So the BIP for case a is

$$\begin{aligned} \max_{x_{11},\ldots,x_{33}} \quad & 100x_{11} + 50x_{12} + 80x_{21} + 40x_{22} + 20x_{31} + 15x_{32} \\ \text{s.t.} \quad & 4x_{11} + 0x_{12} + 4x_{21} + 0x_{22} + x_{31} + 0x_{32} \leq 5 \\ & 200x_{11} + 100x_{12} + 150x_{21} + 90x_{22} + 45x_{31} + 30x_{32} \leq 300 \\ & x_{ij} \in \{0, 1\} \quad \forall i, j \end{aligned} \qquad \text{(1 point for the max)}$$

(6 points if the student directly write down the BIP and it is all correct. Otherwise, deduct the points according to the missing portion.)

Solution Case b The BIP for case b is the same as case a with one additional contraint:

$$\sum_j x_{ij} \leq 1 \quad \forall i \qquad \text{(1 point, 1/6 each)}$$

(Do not accept = sign.)
The BIP is then

$$\begin{aligned} \max_{x_{11},\ldots,x_{33}} \quad & 100x_{11} + 50x_{12} + 80x_{21} + 40x_{22} + 20x_{31} + 15x_{32} \\ \text{s.t.} \quad & 4x_{11} + 0x_{12} + 4x_{21} + 0x_{22} + x_{31} + 0x_{32} \leq 5 \\ & 200x_{11} + 100x_{12} + 150x_{21} + 90x_{22} + 45x_{31} + 30x_{32} \leq 300 \\ & \textstyle\sum_j x_{ij} \leq 1 \quad \forall i \\ & x_{ij} \in \{0, 1\} \quad \forall i, j \end{aligned}$$

(7 points if the student directly write down the BIP and it is all correct. Otherwise, deduct the points according to the missing portion.)

2 Stock trader problem (20 points)

You are a wall street trader, you job is to determine when to buy or sell stock. You are given a price chart showing the price of a list of stocks within a period of time. You want to maximize your net profit by choosing certain days to buy and sell stocks.

There are three rules in stock market:

- You cannot sell a stock if you do not own any.
- You can either buy, sell or do nothing on each day; you cannot buy and sell the same stock on the same day.
- You can only buy or sell **one unit** of stock each time.

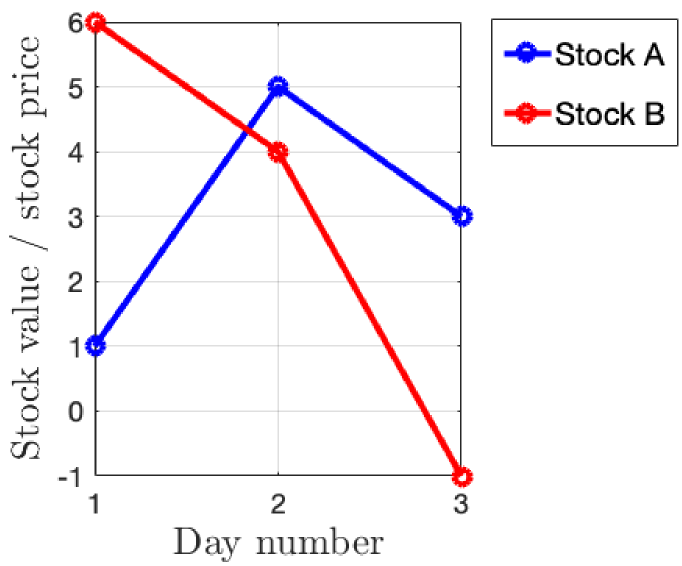
You start with no money. You borrow money from the bank to buy stock, and you have to return the borrowed money (with no interest).

Example For the price graph on the right, an optimal solution is

Day	1	2	3
Action	Buy A	Sell A	Do nothing

Explanation

- You buy 1 unit of stock A at day 1. Now your net gain is -1.
- You sell 1 unit of stock A at day 2. Now your net gain is -1+5 = 4.
- At day 3, as you do not have stock A in your hand, you choose to do nothing.
- For stock B, after viewing the price graph, you decided not to touch it completely (do nothing on stock B for the three days).



Question 1 Formulate the above trading problem on the two stocks as a binary integer program. State clearly and explain your variable(s), your cost function and constraint(s). * Note that the BIP should have the same optimal solution as stated above.

Solution (matrix variable version)

- Decision variables: $X, Y \in \{0,1\}^{3\times 3}$.
  - $X_{ij} = 1$ means we buy stock A at day-i and sell at day-j.
  - $Y_{ij} = 1$ means we buy stock B at day-i and sell at day-j.

  Let $X_{ij} = 0$ for all $i \leq j$ (similarly for Y). It removes the possibility of"selling before buying" and "buy and sell on the same day".

  As a result, we are left with 6 variables instead of 18:

$$X = \begin{bmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & y_{12} & y_{13} \\ 0 & 0 & y_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

- Let $C^k, k = \{A, B\}$ be 3×3 matrices where $c^k_{ij} = p^k_i - p^k_j$. We use k to label stock A or B.

The optimization problem is

$$\begin{aligned} \max_X \quad & \sum_{ij} c^A_{ij} x_{ij} + c^B_{ij} y_{ij} \\ \text{s.t.} \quad & x_{ij} \in \{0,1\},\ y_{ij} \in \{0,1\} \ \ \forall\, i, j \\ & \mathbf{x} \in \mathcal{X} := \Big\{ x_{12} + x_{13} \leq 1,\ x_{13} + x_{23} \leq 1 \Big\} \\ & \mathbf{y} \in \mathcal{Y} := \Big\{ y_{12} + y_{13} \leq 1,\ y_{13} + y_{23} \leq 1 \Big\}. \end{aligned}$$

Explanation: the sets $\mathcal{X}, \mathcal{Y}$ prevent the possibilities of selling stock while holding none of it. For example, the linear inequality constraint $x_{12} + x_{13} \leq 1$ refer to the case that if we buy stock at day 1 and sell it at day 3, we cannot perform x_{12}, because we will not have enough stock to sell.

Solution (vector variable version) If using vector instead of matrix, then need to define four variables

- Decision variables: $x^{\text{buy}}, x^{\text{sell}}, y^{\text{buy}}, y^{\text{sell}} \in \{0,1\}^{3\times 1}$.
  - $x^{\text{buy}}_i = 1$ means we buy stock A at day-i
  - $x^{\text{sell}}_i = 1$ means we sell stock A at day-i
  - $y^{\text{buy}}_i = 1$ means we buy stock B at day-i
  - $y^{\text{sell}}_i = 1$ means we sell stock B at day-i

Let $x^{\text{sell}}_1 = y^{\text{sell}}_1 = 0$. it removes the possibility of"selling before buying" in day 1. As a result, we are left with (3+2)×2 = 10 variables:

$$x^{\text{buy}}_1,\ x^{\text{buy}}_2,\ x^{\text{buy}}_3, x^{\text{sell}}_2,\ x^{\text{sell}}_3, \qquad\qquad y^{\text{buy}}_1,\ y^{\text{buy}}_2,\ y^{\text{buy}}_3, y^{\text{sell}}_2,\ y^{\text{sell}}_3$$

The optimization problem is

$$\begin{array}{lll}
\max\limits_{x^{\text{buy}},x^{\text{sell}},y^{\text{buy}},y^{\text{sell}}} & \sum_{i=1}^{3} p_i^A(x_i^{\text{buy}} - x_i^{\text{sell}}) + p_i^B(y_i^{\text{buy}} - y_i^{\text{sell}}) & \\
\text{s.t.} & x_i^{\text{buy}}, x_i^{\text{sell}}, y_i^{\text{buy}}, y_i^{\text{sell}} \in \{0,1\} \ \ \forall\, i & \\
& x_i^{\text{buy}} + x_i^{\text{sell}} \le 1 \ \ \forall\, i & \text{(cannot buy-and-sell same day on stock A)} \\
& y_i^{\text{buy}} + y_i^{\text{sell}} \le 1 \ \ \forall\, i & \text{(cannot buy-and-sell same day on stock B)} \\
& (x^{\text{buy}}, x^{\text{sell}}) \in \mathcal{X} & \\
& (y^{\text{buy}}, y^{\text{sell}}) \in \mathcal{Y} &
\end{array}$$

where $\mathcal{X}, \mathcal{Y}$ are sets to prevent selling before holding any stock. For example:

- if do not buy in day 1 then cannot sell in day 2

$$x_1^{\text{buy}} \ge x_2^{\text{sell}} \qquad \text{or equivalently} \qquad x_2^{\text{sell}} - x_1^{\text{buy}} \le 0$$

- if do not buy in day 1 then cannot sell in day 3

$$x_1^{\text{buy}} \ge x_3^{\text{sell}} \qquad \text{or equivalently} \qquad x_3^{\text{sell}} - x_1^{\text{buy}} \le 0$$

and similar for other constraints.

Marking : total 8 points

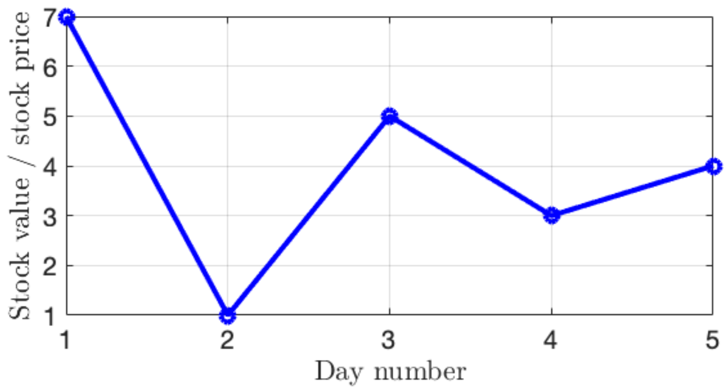
- Correct variables: 2 points
- Correct cost function: 1 point
- Correct 0-1 constraint: 1 point
- Correct constraint to prevent overselling: 4 points
  - 2 for not selling-and-buying in the same day
  - 2 for not selling while not holding any stock

Question 2 Another stock trader focus on a single stock over a longer period of time. The figure below shows the stock price.

Solve the stock trading problem. Write down the optimization problem, state clearly and explain your variable(s), your cost function and constraint(s), and solve the program.

Solution (matrix variable version) Here we show only the matrix version of the solution. Basically the modelling is the same as the first part.

- Let N be a 5×5 matrix where $N_{i,j} = p_i - p_j$.



- Let X be a 5×5 matrix decision variable. $X_{ij} = 1$ means we buy at day-i and sell at day-j. Note that we will let $X_{i,j} = 0$ for all $i \leq j$. This remove the possibility of "selling before buying" and "buy and sell on the same day". As a result, we are left with 10 variables instead of 25:

$$X = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 0 & x_{23} & x_{24} & x_{25} \\ 0 & 0 & 0 & x_{34} & x_{35} \\ 0 & 0 & 0 & 0 & x_{45} \end{bmatrix}$$

With such X, the optimization problem is

$$\max_X \sum_{ij} n_{ij} x_{ij} \quad \text{s.t.} \quad x_{ij} \in \{0,1\} \;\; \forall\, i,j \quad \text{and} \quad \sum_{(i,j) \in \text{neighbour}(i,j)} x_{ij} \leq 1 \; \forall i,j$$

where neighbour(i,j) means

$$\mathcal{N}(i,j) := \Big\{ (p,q) \;\Big|\; |p-i| \leq 1, |q-j| \leq 1 \Big\}$$

For example, on the variable x_{34}, we have the linear inequality constraint

$$x_{14} + x_{24} + x_{34} + x_{35} \leq 1,$$

which means that if we buy stock at day 3 and sell it at day 4, we cannot perform x_{14}, x_{24} and x_{35}, because we do not have enough stock to sell.

Or list all the constraints

$$\begin{array}{l}
x_{12} + x_{13} + x_{14} + x_{15} \leq 1 \\
x_{13} + x_{23} \leq 1 \\
x_{14} + x_{24} + x_{34} \leq 1 \\
x_{15} + x_{25} + x_{35} + x_{45} \leq 1 \\
x_{23} + x_{24} + x_{25} \leq 1 \\
x_{14} + x_{24} + x_{34} \leq 1 \\
x_{34} + x_{35} \leq 1
\end{array}$$

Optimal solution: $x_{23} = x_{45} = 1$, net gain $= (-1+5) + (-3+4) = 5$.

Marking: total 12 points

- Correct variables: 1 point
- Correct cost function: 1 point
- Correct 0-1 constraint: 1 point
- Correct constraint to prevent overselling: 7 points (3.5 + 3.5)
- Correct solution $\mathbf{x}$ 1 point, correct net gain 1 point.

* If the student get the correct solution but not listing all the constraints correctly, they have to show "certain constraints are redundant" or to verify their constraint, otherwise deduct 3.5 points for possibilities of infeasible solution.

END of assignment.