Solution to CO327 (2022Spring) Assignment 5

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1 Making dogs happy / Welfare policy (11 points)

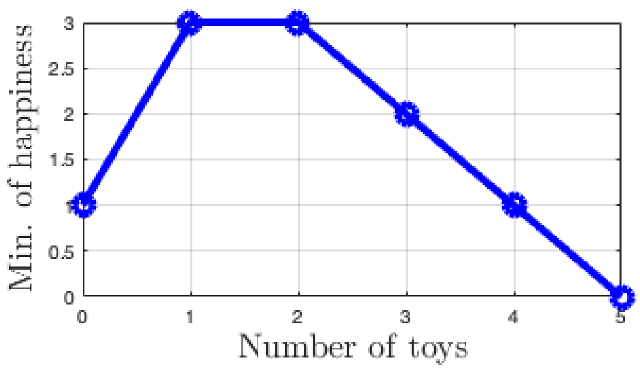
You have 3 dogs (D1, D2, D3) and you give toys to them. Since you are a busy person, you have decided to give the same number of toys to each dog. Let x be the number of toys each dog receives. As you are very familiar with the dogs, you know the happiness level of each dog as a function of the number of toy it receives:

$$D1 : 1 + 2x, \quad D2 : 2 + x, \quad D3 : 5 - x.$$

You have a budget limit that you can give each dog at most 5 toys. To be fair to the dogs, you want to maximize the minimum happiness level of all the dogs.

1 Let $f(x)$ be the minimum happiness level of all 3 dogs when each dog receives x toys. Write down the expression of $f(x)$. Draw $f(x)$ for $x \in \{0, 1, 2, 3, 4, 5\}$.

$$f(x) = \min\Big\{1 + 2x, 2 + x, 5 - x\Big\}. \qquad \text{(1 point)}$$



Correct figure 1 point. If no figure no point.

2 Translate this problem to an optimization problem. Solve for the optimal x and list the happiness level of the dogs.

$$\begin{array}{ll} \max\limits_{x} & \min\Big\{1 + 2x, 2 + x, 5 - x\Big\} \\ \text{s.t.} & 0 \leq x \leq 5,\ x \in \mathbb{N} \end{array} \qquad \text{(3 points)}$$

Optimal solution $x^* = 1$ or 2 (non-unique) 1 point

Happiness level: $3, 3, 2$ 1 point

Careless mistake If student make a mistake that the objective function is $1 + 2x + 2 + x + 5 - x$, then whole part no point (all 5 points gone)

3 Now, instead of maximizing the minimum happiness level of all 3 dogs, you decided to maximize the **sum** of the happiness factors of all 3 dogs. Write down the optimization problem. What is the optimal x? What are the happiness level of the dogs?

$$\begin{aligned} \max_{x} \quad & 1+2x+2+x+5-x = 8+2x \\ \text{s.t.} \quad & 0 \le x \le 5,\ x \in \mathbb{N} \end{aligned} \qquad \text{(1 point)}$$

Optimal $x = 5$ 1 point
Happiness level: $11, 7, 0$ 1 point

4 Consider give toys to dogs as an analogy to deciding a welfare policy. Now "toys" are welfare support and "dogs" are people. Comparing the approaches of (2) and (3) and their solution, which approach you prefer? Why?

(2), more fair / more uniform performance than maximizing the sum. 1 point
(If student pick (3), accept all reasonable answer, otherwise at most 0.5 point)

Careless mistake If student previously make a mistake that approach in (2) and (3) have the same objective function, this part get no point

2 Scheduling problem (13 points)

You have a train company and you want to determine the arrival time of 3 trains. Let t_i be the arrive time of the ith train, the time t_i are arranged in ascending order $0 \le t_1 \le t_2 \le t_3 \le +\infty$. However, for safety reason, you want to maximize the smallest time gap between *any two consecutive trains*, subject to time range constraints $l_i \le t_i \le u_i$, $i = 1, 2, 3$, where

i	1	2	3
l_i	0	1	2
u_i	2	3	4

Write down the optimization problem in *canonical form*. Solve this optimization problem. If the problem has no solution, explain why.

When you constructing the constants $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$, the code

- `zeros(h,k)` will create a all-zero-matrix of size h-by-k
- `ones(h,k)` will create a all-one-matrix of size h-by-k
- `zeros(h,1)` will create a zero column vector of size h-by-1
- `ones(h,1)` will create a one column of size h-by-1
- `eye(n)` will create a n-by-n identity matrix
- the code `[u v]` will stack two column vectors $\boldsymbol{u}, \boldsymbol{v}$ horizontally to form a matrix $[\boldsymbol{u}\ \boldsymbol{v}]$
- the code `[u; v]` will stack two column vectors $\boldsymbol{u}, \boldsymbol{v}$ vertically to form a vector $[\boldsymbol{u}^\top\ \boldsymbol{v}^\top]^\top$

Sol The max-min expression

$$\begin{array}{ll} \max & \left(\min_i (t_{i+1} - t_i)\right) \\ \text{s.t.} & l_i \leq t_i \leq u_i, i \in \{1,2,3\} \end{array}$$

Using the worst-case modeling trick,

$$\text{let } \Delta = \min_i (t_{i+1} - t_i),$$

the equivalent LP in canonical form is

$$\max_{\boldsymbol{t},\Delta} \begin{bmatrix} \mathbf{0}_{3\times1} \\ 1 \end{bmatrix}^\top \begin{bmatrix} \boldsymbol{t}_{3\times1} \\ \Delta \end{bmatrix} \text{ s.t. } \begin{bmatrix} \boldsymbol{I}_{3\times3} & \mathbf{0}_{3\times1} \\ -\boldsymbol{I}_{3\times3} & \mathbf{0}_{3\times1} \\ \boldsymbol{D}_{2\times3} & \mathbf{1}_{2\times1} \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \Delta \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{u}_{3\times1} \\ -\boldsymbol{l}_{3\times1} \\ \mathbf{0}_{2\times1} \end{bmatrix}$$

where $\boldsymbol{D}$ is the first-order difference operator :

$$\boldsymbol{D} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Marking on the problem formulation

- Correct $\boldsymbol{c}$ 2 points
- Correct $\boldsymbol{A}$ 6 points
- Correct $\boldsymbol{b}$ 3 points

MATLAB The code for the above formulation is

```
c=[ zeros(3,1); 1]
D = [1 -1 0; 0 1 -1]
A = [ [eye(3) zeros(3,1)]; [-eye(3) zeros(3,1)]; [D ones(2,1)] ]
b = [2 3 4 0 -1 -2 0 0]'
```

Note that this is a max problem, so we need to call `td = linprog(-c,A,b);` (1 pt), which gives

$$\texttt{td} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix}$$

with smallest time gap 2. 2 pts for $\boldsymbol{t}$ and 1 pts for time gap

Careless mistake If wrong sign (e.g. a negative sign in the time gap or in the time), -1 point each

END of assignment.