# Solution to CO327 (2022Spring) Assignment 7 - Image completion

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The important trick here: see midterm Q3.2 !

$$\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|_1 \quad \Longleftrightarrow \quad \min_{\mathbf{x},\mathbf{u}} \mathbf{1}^\top \mathbf{u} \text{ s.t. } -\mathbf{u} \le \mathbf{A}\mathbf{x} \le \mathbf{u} \qquad \text{(Important trick)}$$

## (*)

Let the size of $\mathbf{E}$ be $M \times N$, then

$$\min \|\mathbf{E}\mathbf{x}\|_1 \text{ s.t. } \mathbf{S}\mathbf{x} = \hat{\mathbf{x}}_\Omega,\ \mathbf{x} \ge \mathbf{0}, \quad \Longleftrightarrow \quad \min \hat{\mathbf{c}}^\top \hat{\mathbf{x}} \text{ s.t. } \hat{\mathbf{A}}\hat{\mathbf{x}} \le \hat{\mathbf{b}},\ \hat{\mathbf{A}}_{\text{eq}}\hat{\mathbf{x}} = \hat{\mathbf{b}}_{\text{eq}},\ \hat{\mathbf{x}} \ge \mathbf{0}$$

where $\hat{\mathbf{x}} = \begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} \in \mathbb{R}^{(n^2+M)\times 1}$, $\hat{\mathbf{c}} = \begin{bmatrix}\mathbf{0}_{n^2\times 1}\\ \mathbf{1}_{M\times 1}\end{bmatrix} \in \mathbb{R}^{(n^2+M)\times 1}$, and

$$\hat{\mathbf{A}} = \begin{bmatrix}\mathbf{E} & -\mathbf{I}_M\\ -\mathbf{E} & -\mathbf{I}_M\end{bmatrix},\ \hat{\mathbf{b}} = \mathbf{0}_{2M\times 1},\ \hat{\mathbf{A}}_{\text{eq}} = \begin{bmatrix}\mathbf{S} & \mathbf{0}_{|\Omega|\times M}\end{bmatrix},\ \hat{\mathbf{b}}_{\text{eq}} = \hat{\mathbf{x}}_\Omega$$

$(p, q)$

$$\min \begin{bmatrix}\mathbf{0}_{n^2}\\ \mathbf{1}_{M\times 1}\\ \mathbf{1}_{|\Omega|\times 1}\end{bmatrix}^\top \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\\ \mathbf{z}\end{bmatrix} \text{ s.t. } \begin{bmatrix}-\mathbf{S} & \mathbf{0}_{|\Omega|\times M} & -\mathbf{I}_{|\Omega|}\\ \mathbf{S} & \mathbf{0}_{|\Omega|\times M} & -\mathbf{I}_{|\Omega|}\\ -\mathbf{E} & -\mathbf{I}_M & \mathbf{0}_{M\times|\Omega|}\\ \mathbf{E} & -\mathbf{I}_M & \mathbf{0}_{M\times|\Omega|}\end{bmatrix} \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\\ \mathbf{z}\end{bmatrix} \le \begin{bmatrix}-\hat{\mathbf{x}}_\Omega\\ \hat{\mathbf{x}}_\Omega\\ \mathbf{0}_{|\Omega|\times 1}\\ \mathbf{0}_{|\Omega|\times 1}\end{bmatrix} \qquad (1,1)$$

$$\min \begin{bmatrix}\mathbf{0}_{n^2}\\ \mathbf{1}_{M\times 1}\\ 1\end{bmatrix}^\top \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\\ t\end{bmatrix} \text{ s.t. } \begin{bmatrix}-\mathbf{S} & \mathbf{0}_{|\Omega|\times M} & -\mathbf{1}_{|\Omega|\times 1}\\ \mathbf{S} & \mathbf{0}_{|\Omega|\times M} & -\mathbf{1}_{|\Omega|\times 1}\\ -\mathbf{E} & -\mathbf{I}_M & \mathbf{0}_{M\times 1}\\ \mathbf{E} & -\mathbf{I}_M & \mathbf{0}_{M\times 1}\end{bmatrix} \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\\ t\end{bmatrix} \le \begin{bmatrix}-\hat{\mathbf{x}}_\Omega\\ \hat{\mathbf{x}}_\Omega\\ \mathbf{0}_{|\Omega|\times 1}\\ \mathbf{0}_{|\Omega|\times 1}\end{bmatrix} \qquad (1,\infty)$$

$$\min \begin{bmatrix}\mathbf{0}_{n^2}\\ 1\\ \mathbf{1}_{|\Omega|\times 1}\end{bmatrix}^\top \begin{bmatrix}\mathbf{x}\\ s\\ \mathbf{z}\end{bmatrix} \text{ s.t. } \begin{bmatrix}-\mathbf{S} & \mathbf{0}_{|\Omega|\times 1} & -\mathbf{I}_{|\Omega|}\\ \mathbf{S} & \mathbf{0}_{|\Omega|\times 1} & -\mathbf{I}_{|\Omega|}\\ -\mathbf{E} & -\mathbf{1}_{M\times 1} & \mathbf{0}_{M\times|\Omega|}\\ \mathbf{E} & -\mathbf{1}_{M\times 1} & \mathbf{0}_{M\times|\Omega|}\end{bmatrix} \begin{bmatrix}\mathbf{x}\\ s\\ \mathbf{z}\end{bmatrix} \le \begin{bmatrix}-\hat{\mathbf{x}}_\Omega\\ \hat{\mathbf{x}}_\Omega\\ \mathbf{0}_{|\Omega|\times 1}\\ \mathbf{0}_{|\Omega|\times 1}\end{bmatrix} \qquad (\infty,1)$$

$$\min \begin{bmatrix}\mathbf{0}_{n^2}\\ 1\\ 1\end{bmatrix}^\top \begin{bmatrix}\mathbf{x}\\ s\\ t\end{bmatrix} \text{ s.t. } \begin{bmatrix}-\mathbf{S} & \mathbf{0}_{|\Omega|\times 1} & -\mathbf{1}_{|\Omega|\times 1}\\ \mathbf{S} & \mathbf{0}_{|\Omega|\times 1} & -\mathbf{1}_{|\Omega|\times 1}\\ -\mathbf{E} & -\mathbf{1}_{M\times 1} & \mathbf{0}_{M\times 1}\\ \mathbf{E} & -\mathbf{1}_{M\times 1} & \mathbf{0}_{M\times 1}\end{bmatrix} \begin{bmatrix}\mathbf{x}\\ s\\ t\end{bmatrix} \le \begin{bmatrix}-\hat{\mathbf{x}}_\Omega\\ \hat{\mathbf{x}}_\Omega\\ \mathbf{0}_{|\Omega|\times 1}\\ \mathbf{0}_{|\Omega|\times 1}\end{bmatrix} \qquad (\infty,\infty)$$

where $\mathbf{I}_a$ is identity matrix of $a \times a$ and $\mathbf{I}_{a\times b}$ is identity matrix of $a \times b$ by chopping rows or adding zero rows based on $a, b$.

**Marking** The distribution of %

- Correct recovered solution / figure 10%
- Correct model formulation (if figure is wrong) 30%
- PDF explanation and discussion 10%
  Example: discuss why $(\infty, \infty)$ is the worst one in $(p, q)$

END of assignment.