CO327 (2022Spring) Assignment 8 - Theory & $\LaTeX$

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- Assignment deadline: July-17 23:55.
- Submit your electronic copy (a single PDF) to the dropbox in Waterloo LEARN.
- **Write down your solutions using $\LaTeX$. You get no point if you submit handwritten solution.**

1 Geometry (14 points)

1. Polytopes (9 points) Draw the polytope P. Label each halfspace in your figure and mark down the extreme points (write down their coordinate).

$$P \;:\; \begin{array}{rcl} x_1 + x_2 + x_3 & \leq & 4 \\ x_1 & \leq & 2 \\ x_3 & \leq & 3 \\ 3x_2 + x_3 & \leq & 6 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \\ x_3 & \geq & 0 \end{array}$$

Note: you can draw the polytope by hand and take photo of it. To attach a figure in $\LaTeX$, put

```
\usepackage{graphicx}
```

in the code and use to attach figure.

```
\begin{figure}[t]
\includegraphics[width=8cm]{Plot}
\centering
\end{figure}
```

See `https://www.overleaf.com/learn/latex/Inserting_Images` for more information.

2. Hyperplane (4 points) Copy the figure you drawn in the previous part. Now draw the direction vector (the normal vector) of the hyperplane of the first 4 inequalities in P.

3. Maximization over a polytope (1 point) Consider the LP:

$$\max_{x_1,x_2,x_3} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{s.t. } \mathbf{x} \in P.$$

Based on

- the figures you drawn, and;
- the fact that at a solution point $\mathbf{x}^*$, the vector $\mathbf{c}$ with be inside the cone generated by the direction of the active hyperplane of $\mathbf{x}^*$,

deduce where is the solution of this LP without calling MATLAB.

2 Simple LP (12 points)

Consider problem $\mathcal{P}$:

$$\mathcal{P} \;:\; \begin{array}{ll} \max\limits_{x_1,x_2} & 7x_1 + 5x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 100 \\ & 4x_1 + 3x_2 \leq 240 \\ & x_1 \geq 0, x_2 \geq 0 \end{array} \;, \quad \text{with solution} \;\; (x_1^*, x_2^*) = (30, 40).$$

1. (5 points) Write down the dual problem $\mathcal{D}$ of $\mathcal{P}$ in the symmetric dual form.

2. (1 point) Solve the dual problem.
Hint Be-careful with the direction of the inequalities when using MATLAB.

3. (2 points) Verify the weak duality.

4. (4 points) Verify the complementary slackness of the solution $\mathbf{x}^*, \mathbf{y}^*$ in $\mathcal{P}$ and $\mathcal{D}$.
Hint What you need to do is to identify

- which slack constraint in the primal correspond to which tight constraint in the dual, and
- which tight constraint in the primal correspond to which slack constraint in the dual.

3 Duality of LP (12 points)

Notation In this question,

- $\mathbf{x}^*$ denotes the optimal solution of the primal problem .
- $p^* = \mathbf{c}^\top \mathbf{x}^*$ denotes the optimal objective function value of the primal problem.
- $\mathbf{y}^*$ denotes the optimal solution of the dual problem.
- $d^* = \mathbf{b}^\top \mathbf{y}^*$ denotes the optimal objective function value of the dual problem.

The problem Consider the following LP of single variable: maximize cx subject to $ax \leq b$ and $x \geq 0$ with $a = 0, c = b = 1$. That is,

$$(P) \begin{cases} \max\limits_{x} & 1 \cdot x \\ \text{s.t.} & 0 \cdot x \leq 1 \\ & x \geq 0 \end{cases} \qquad (D) \begin{cases} \min\limits_{y} & 1 \cdot y \\ \text{s.t.} & 0 \cdot y \geq 1. \\ & y \geq 0 \end{cases}$$

Alice saw problem (P), she wrote down the symmetric dual problem (D) as shown above. She then solves both problems and found that $x^* = p^* = +\infty$, and $y^* = d^* = 0$. She found that $p^* \not\leq d^*$ and said "I have proved that weak duality is wrong!". Comment on her conclusion. If she is correct, explain why. If she is incorrect, explain where is the error and give the correct result.

END of assignment.