Solution to CO327 (2022Spring) Assignment 8 - Theory

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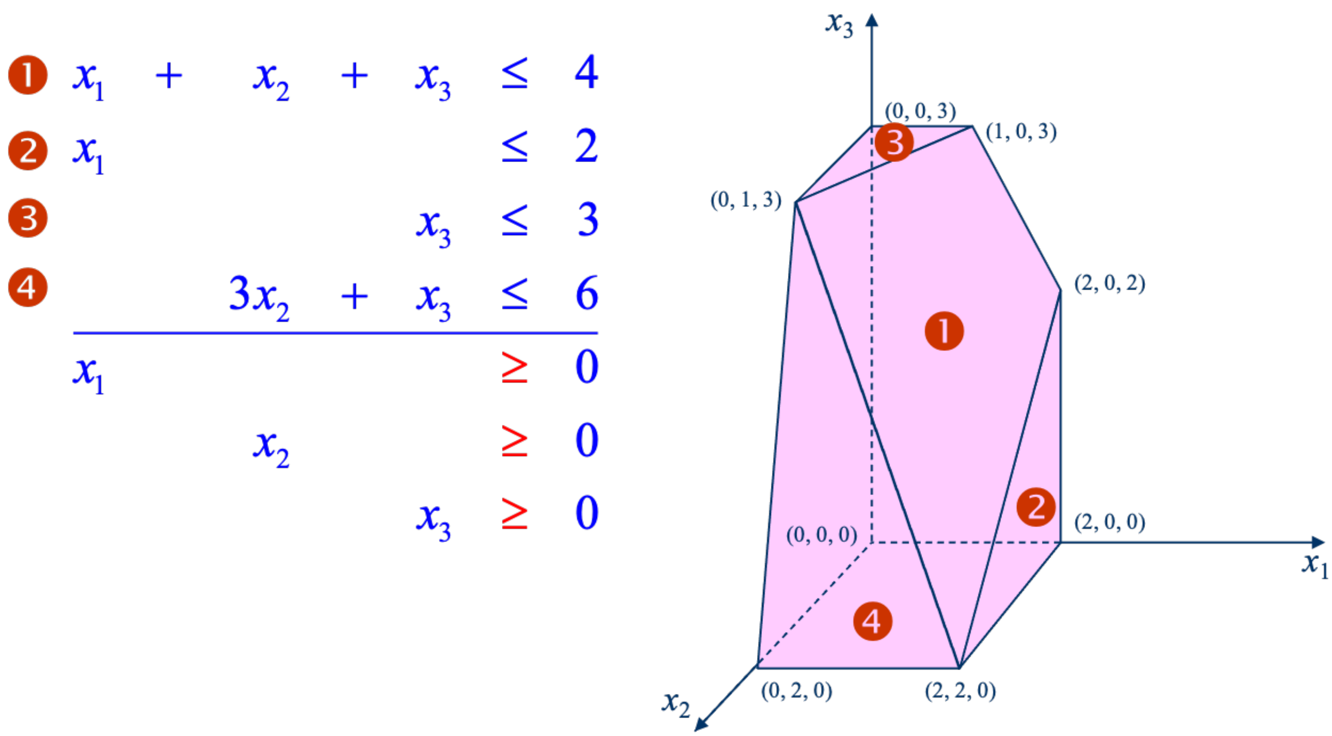
1 Geometry (14 points)

1. Polytopes (9 points) Draw the polytope P. Label each halfspace in your figure and also mark the extreme points in the figure (write down their coordinate).

$$P : \quad \begin{array}{rcl} x_1 + x_2 + x_3 & \leq & 4 \\ x_1 & \leq & 2 \\ x_3 & \leq & 3 \\ 3x_2 + x_3 & \leq & 6 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \\ x_3 & \geq & 0 \end{array}$$

Solution Ineq 1-4 (1 point each), $x_i \geq 0$ (1 point), extreme points 0.5 each.



2. Hyperplane (4 points) Copy the figure you drawn in the previous part. Now draw the direction vector (the normal vector) of the hyperplane of the first 4 inequalities in P.

3. Maximization over polytope (1 point) Consider the LP:

$$\max_{x_1,x_2,x_3} \begin{bmatrix}2\\0\\1\end{bmatrix}^\top \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \quad \text{s.t. } \mathbf{x} \in P.$$

Based on

- the figure you drawn, and;
- the fact that at the solution point, the vector **c** with be inside the cone generated by the direction of the active hyperplane,

deduce where is the solution of this LP.

Solution (2,0,2)

2 Simple LP (12 points)

Consider the following problem:

$$\mathcal{P} : \begin{array}{ll} \max\limits_{x_1,x_2} & 7x_1 + 5x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 100 \\ & 4x_1 + 3x_2 \leq 240 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}, \quad \text{with solution } (x_1^*, x_2^*) = (30, 40).$$

1. Dual problem (5 points) Write down the dual problem $\mathcal{D}$ of $\mathcal{P}$ in the symmetric dual form.

$$\mathcal{D} : \begin{array}{ll} \min\limits_{y_1,y_2} & 100y_1 + 240y_2 \\ \text{s.t.} & 2y_1 + 4y_2 \geq 7 \\ & y_1 + 3y_2 \geq 5 \\ & y_1 \geq 0, y_2 \geq 0 \end{array}$$

min (1 point), $b, \mathbf{A}^\top, \mathbf{c}$ 1 point each, and $\mathbf{y} \geq \mathbf{0}$ 1 point

2. (1 point) Solve the dual problem. **Hint: be-careful of the direction of the inequality when using MATLAB.**

```
clear,close all,
c = [7 5]';
b = [100 240]';
A = [2 1 ; 4 3];
[x, xfval] = linprog(-c,A,b,[],[],[0;0]);
[y, yfval] = linprog(b,-A',-c,[],[],[0;0]);
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Figure 1: $\mathbf{y}^* = (0.5, 1.5)$ (1 point)

3. (2 points) Verify the weak and strong duality.

$p^* = 410 = d^*$.

4. (4 points) Verify the complementary slackness of the solution $\mathbf{x}^*, \mathbf{y}^*$ in $\mathcal{P}$ and $\mathcal{D}$. What you need to do: identify

- which slack constraint in the primal correspond to which tight constraint in the dual, and
- which tight constraint in the primal correspond to which slack constraint in the dual

With $\mathbf{x}^* = \begin{bmatrix} 30 \\ 40 \end{bmatrix}$, $\mathbf{y}^* = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$,

$$\begin{aligned}
\text{primal variable 1 is slack } x_1^* > 0 \quad &\iff \quad \text{dual constraint 1 is tight } 2y_1^* + 4y_2^* = 7 \\
\text{primal variable 2 is slack } x_2^* > 0 \quad &\iff \quad \text{dual constraint 2 is tight } y_1^* + 3y_2^* = 5 \\
\text{dual variable 1 is slack } y_1^* > 0 \quad &\iff \quad \text{primal constraint 1 is tight } 2x_1^* + x_2^* = 100 \\
\text{dual variable 2 is slack } y_2^* > 0 \quad &\iff \quad \text{primal constraint 2 is tight } 4x_1^* + 3x_2^* = 240
\end{aligned}$$

3 Duality of LP (12 points)

Notation In this question,

- $\mathbf{x}^*$ denotes the optimal solution of the primal problem
- $p^* = \mathbf{c}^\top \mathbf{x}^*$ denotes the optimal primal cost value / optimal objective function value of the primal problem
- $\mathbf{y}^*$ denotes the optimal solution of the dual problem
- $d^* = \mathbf{b}^\top \mathbf{y}^*$ denotes the optimal dual cost value / optimal objective function value of the dual problem

The problem Consider the following LP of single scalar variable: maximize cx subject to $ax \leq b$ and $x \geq 0$ with $a = 0, c = b = 1$. That is,

$$(P) \begin{cases} \max\limits_x & 1 \cdot x \\ \text{s.t.} & 0 \cdot x \leq 1. \\ & x \geq 0 \end{cases} \qquad (D) \begin{cases} \min\limits_y & 1 \cdot y \\ \text{s.t.} & 0 \cdot y \geq 1. \\ & y \geq 0 \end{cases}$$

A student Alice saw this problem (P), she wrote down the symmetric dual problem (D) as shown above. She then solved both problems and found that $x^* = p^* = +\infty$, $y^* = d^* = 0$. She found that $p^* \not\leq d^*$ and said "I have proved that weak duality is wrong!". Comment on her conclusion. If she is correct, explain why. If she is incorrect, explain where is the error and give the correct result.

Alice is wrong. 1 pt.

The constraint $0 \cdot x \leq 1$ have to be removed from (P) 1 pt.

therefore

$$(P) \begin{cases} \max\limits_x & 1 \cdot x \\ \text{s.t.} & x \geq 0 \end{cases} \iff \begin{cases} \max\limits_x & 1 \cdot x \\ \text{s.t.} & -x \leq 0 \end{cases} \iff \begin{cases} \max & \begin{bmatrix} 1 \\ -1 \end{bmatrix}^\top \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \\ \text{s.t.} & \\ & [-1 \ 1] \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \leq 0 \\ & \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \geq 0 \end{cases} \qquad \text{(2 pt.)}$$

This (P) is unbounded therefore thee corresponding symmetric dual (D) is infeasible, which is true: 2+2 pt.

$$(D)\begin{cases} \min & 0\cdot y \\ \text{s.t.} & \begin{bmatrix}-1\\1\end{bmatrix} y \geq \begin{bmatrix}1\\-1\end{bmatrix} \\ & y \geq 0 \end{cases}$$

there is no y such that (D) is feasible. 2 pt.

Note: full marks only give to those who correctly reformulate (P) and (D). If only saying "(D) is infeasible" without showing the correct reformulation will only get 2 pts most.

END of assignment.