CO327 (2022Spring) Final assignment

Lecturer: Andersen Ang

July 18, 2022

Instructions

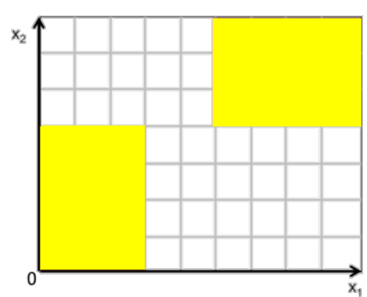
- Answer ALL questions.
- Submit your solution in **one single PDF** (no need to submit MATLAB code), write down your name and student ID. **You get zero points if you submit png or jpg.**
- LaTeX is not necessary.
- Submit your PDF to the dropbox in Waterloo LEARN before the deadline July-31 23:55 (EDT).
  - **You get zero point for late submission.**
  - If you re-submit, only the newest submission will be considered and your previous submission(s) will be ignored.
- Use of MATLAB is allowed. **DO NOT** submit the MATLAB code.

Course evaluation

Go to `https://perceptions.uwaterloo.ca/` or `http://evaluate.uwaterloo.ca` to fillin the course survey of CO 327. **Do this before Tuesday July 26, 11:59 p.m.**

1 One set of inequalities for two feasible regions (10 points)

Consider the following figure.



In the figure, the gird starts with 0 at the lower bottom left. There are two yellow regions, representing the feasible region of a particular linear program. The first yellow region is a rectangle with the lower left corner at (0,0) and the upper right corner (3,4). The first yellow region is a rectangle with the lower left corner at (5,4) and the upper right corner (9,7).

Write down **one set of inequalities** to represent the **two** yellow feasible regions.

2 NBA manager (20 points)

NBA is a professional basketball league in North America. You want to build a new team called "Waterloo Wolf".

Players According to NBA regulations, the number of players of a basketball team on the court is 5. That is, your team has at least 5 players. You can hire more than 5 players, those extra players are called backups.

Wages You hire players to form a team. You select players from the market. There are n available players $i \in \{1, 2, \ldots, n\}$ with wages $w_i \geq 0$. The nonnegative number w_i represents the wages of player i. If you hire player i, you have to pay w_i to i as salary.

Sponsorship Your own startup fund is not enough for the wages, so you look for investor. There are m investors $j \in \{1, 2, \ldots, m\}$, each provides sponsorship $s_j \geq 0$ to you, that is, give you s_j amount of money to support your NBA team. However, there is a condition: the investor $j \in \{1, 2, \ldots, m\}$ will give you the sponsorship s_j only if certain players $P_j \subseteq \{1, 2, \ldots, n\}$ are included in the team, because the investors want these players to do ads for them in the future. The set P_j indicates which player i the investor j want to be hired in your team, and all player $i \in P_j$ must be chosen in order to receive funding s_j from investor j.

Profit As a NBA manager, your profit is the sum of the sponsorship money from investors minus the salary (sum of payments to players).

Example Here is an example with $n = 4$ and $m = 5$

Available player i	w_i
1	100
2	80
3	120
4	150

Investor j	s_j	P_j
1	120	1,3
2	300	1,3,4
3	240	3,4
4	180	4
5	65	2

In this example, there are 4 players available: $i \in \{1, 2, \ldots, 4\}$ and 5 investors: $j \in \{1, 2, \ldots, 5\}$. In the left table, w_3 refers to the wages of player $i = 3$, which is 120\$. In the right table, to get the sponsor from investor $j = 1$, which is $s_1 = 120$\$, your need to hire player $i = 1$ and $i = 3$. Now, suppose hire player 4, then you get sponsorship 4 and your profit will be $180 - 150 = 30$\$.

2.1 Modeling (9 points)

To summarize, you are given

- A number n, the number of available players in the market.
- A list of numbers $w_1, w_2, \ldots, w_n$, referring to the wages of each player
- A number m, the number of investors.
- A list of numbers $s_1, s_2, \ldots, s_m$, referring to the sponsorship of each investor.
- A list of sets $P_1, P_2, \ldots, P_m$, referring to the condition from each investor that which player i has to be included in the team
- Basketball rule: at least 5 players in a team.

Your goal is to maximize your profit. Model this problem (not the example above) as an linear integer program. State clearly your variables, objective function and constraints.

2.2 Solve the NBA management problem (11 points)

After running the team for several months, in a new season your NBA team now has 3 very strong players, so you look for **exactly 2** new players. Solve this NBA management problem using the values in the example above. Which players should you hire? Which investment(s) will you receive? What is your profit?

3 Minesweeper (20 points)

Introduction Given a board containing hidden "mines", your task is to guess where are the mines using the game rule and clues. The figure on the right shows an example of a typical minesweeper game.

If you have never played minesweeper, the best way to understand the game is to try it. You can play it online at https://minesweeperonline.com

Example Consider a 4 4 board shown below. Let the variable be a binary matrix $X = x_{ij}$, where $x_{ij} = 1$ means square-(i, j) has a mine, while $x_{ij} = 0$ means the square is safe. Here (i, j) means $(row, column)$.

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

Solving a minesweeper problem corresponds to finding the correct x_{ij} value subject to the constraints of the game. Your task is to convert minesweeper problem into an integer program.

Game rule On each mined square, there is a number, telling how many mines are presented in the "neighbohood": north, north east, east, south east, south, south west, west and north west. A blank square with no number indicates there is no mine in its neighborhood.

Example: deducing the solution The player solves the variable X by deduction using the game rule and the hints. Mathematically we uncover the unknown X as

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 0 & 0 & 0 & x_{14} \\ 0 & 0 & 0 & x_{24} \\ 0 & 0 & 0 & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \xrightarrow{2,3,4,5} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where 1,2,3,4,5 are:

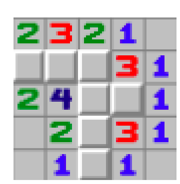
1. The squares (1,1), (1,2), (2,1), (2,2) have already been played, and have no mine in their respective surrounding squares, these $x_{ij} = 0$.
2. The squares (3,1), (3,2) have been played, and are surrounded by one mined square each (indicated by the number 1). From these we can deduce that (4,1) has a mine.
3. The squares (1,3) and (3,3) each have two mines in their neighborhood.
4. The square (2,3) has three mines around it. We can deduce that (1,4), (2,4) and (3,4) have a mine (indicated by the flag);
5. Finally, based on the game rule and the deduction steps 2-4, we can deduce that (4,2), (4,3) and (4,4) are safe move.

3.1 Modeling the deduction process (5 points)

Formulate the minesweeper problem in the example above as an integer program, using the variable **X** defined above. (You do not need to solve the program, but solving it can check whether or not your modelling is correct)

3.2 Solving a minesweeper problem (15 points)

For the figure on the right, formulate this minesweeper problem as an integer program and solve it.



4 (Modified) Google HashCode 2020 (20 points)

Book scanning

Problem statement for the Online Qualification Round of Hash Code 2020

Introduction

Books allow us to discover fantasy worlds and better understand the world we live in. They enable us to learn about everything from photography to compilers... and of course a good book is a great way to relax!

Google Books is a project that embraces the value books bring to our daily lives. It aspires to bring the world's books online and make them accessible to everyone. In the last 15 years, Google Books has collected digital copies of 40 million books in more than 400 languages[1], partly by scanning books from libraries and publishers all around the world.

In this competition problem, we will explore the challenges of setting up a scanning process for millions of books stored in libraries around the world and having them scanned at a scanning facility.

Task

Given a description of libraries and books available, plan which books to scan from which library to maximize the total score of all scanned books, taking into account that each library needs to be signed up before it can ship books.

Problem description

Books

There are ***B*** different books with IDs from 0 to ***B***–1. Many libraries can have a copy of the same book, but we only need to scan each book once. Each book is described by one parameter: the score that is awarded when the book is scanned.

Libraries

There are ***L*** different libraries with IDs from 0 to ***L***–1. Each library is described by the following parameters:

- the set of **books** in the library,
- the **time** in days that it takes to sign the library up for scanning,
- the **number of books that can be scanned each day** from the library once the library is signed up.

Time

There are ***D*** days from day 0 to day ***D***–1. The first library signup can start on day 0. ***D***–1 is the last day during which books can be shipped to the scanning facility.

Library signup

Each library has to go through a **signup process** before books from that library can be shipped. Only one library at a time can be going through this process (because it involves lots of planning and on-site visits at the library by logistics experts): the signup process for a library can start only when no other signup processes are running. The libraries can be signed up **in any order**.

Books in a library can be scanned as soon as the signup process for that library completes (that is, on the first day immediately after the signup process, see the figure below). Books can be scanned in parallel from multiple libraries.

For **example**, if

- the signup process for library 0 (that is, the library with ID 0) takes 2 days, and
- the signup process for library 1 takes 3 days, and
- library 1 is signed up before library 0

then

- the signup process of library 1 starts on **day 0**, and finishes on **day 2** (3 days in total)
- the first books from library 1 can be scanned starting on **day 3** (the next day after the signup process finishes)
- the signup of library 0 starts on **day 3** (the next day after the signup process of library 0 is done) and finishes on **day 4** (2 days in total)
- the first books from library 0 can be scanned starting on **day 5** (the next day after the signup process finishes)

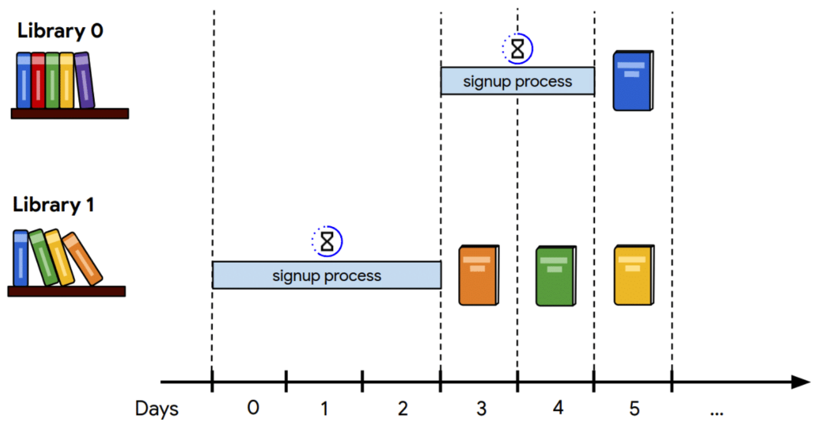


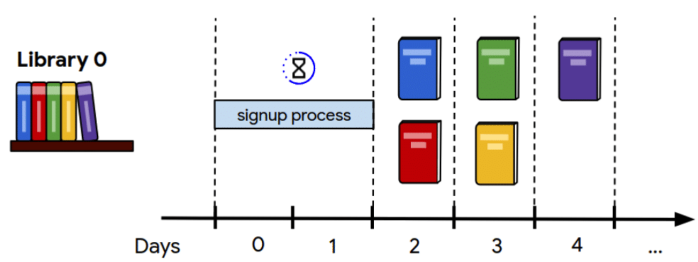
Figure. The signup process (blue bars) and the books being scanned from each library per day. Only one library can be in the signup process at a time, but books can be shipped in parallel by multiple libraries.

Scanning

All books are scanned in the scanning facility. The entire process of sending the books, scanning them, and returning them to the library happens in one day (note that each library has a maximum number of books that can be scanned from this library per day). The scanning facility is big and can scan any number of books per day.

For **example**, if library 0 has 5 books, can ship 2 books per day, and completes the signup process on day 1, then:

- 2 books can be scanned on day 2
- 2 books can be scanned on day 3
- the one remaining book can be scanned on day 4



Input data set

~~The first line of~~ the data set contains:

- an integer $\boldsymbol{B}$ ($1 \le \boldsymbol{B} \le 10^5$) – the number of different books,
- an integer $\boldsymbol{L}$ ($1 \le \boldsymbol{L} \le 10^5$) – the number of libraries,
- an integer $\boldsymbol{D}$ ($1 \le \boldsymbol{D} \le 10^5$) – the number of days.

This is followed by one line containing **B integers:** $\boldsymbol{S_0}, \ldots, \boldsymbol{S_{B-1}}$, ($0 \le \boldsymbol{S_i} \le 10^3$), describing the score of individual books, from book 0 to book **B**–1.

This is followed by **L** sections describing individual libraries from library 0 to library **L**–1. Each such section contains two lines:

- the first line, which contains:
  - $\boldsymbol{N_j}$ ($1 \le \boldsymbol{N_j} \le 10^5$) – the number of books in library ***j***,
  - $\boldsymbol{T_j}$ ($1 \le \boldsymbol{T_j} \le 10^5$) – the number of days it takes to finish the library signup process for library ***j***,
  - $\boldsymbol{M_j}$ ($1 \le \boldsymbol{M_j} \le 10^5$) – the number of books that can be shipped from library ***j*** to the scanning facility per day, once the library is signed up.
- the second line, which contains $\boldsymbol{N_j}$ integers, describing the IDs of the books in the library. Each book ID is listed at most once per library.

The total number of books in all libraries does not exceed 10^6.

4.1 Modelling the book scanning problem (13 points)

As a summary, you are given

- A number B 1, referring to the number of different books if it is scanned
  - Each book has a number s_i 0, referring to the score of that book.
  - There are B books so we have a list of scores $s_1, s_2, \ldots, s_B$.
  - Books are somehow assigned to libraries, one book can be in many libraries.
- A number L 1, referring to the number of libraries. Each library has
  - A set L_j $f1, 2, \ldots, Bg$, indicate which books is present in library j
  - a number N_j 1, referring to the number of books in library j
  - a number T_j 1, referring to the signup time
  - a number M_j 1, referring to the maximum number of books shipped per day
- A number D 1, referring to the number of days to finish the scanning task.

A scanning rules

- We can only choose one library, wait for its signup time, then choose another and so on but all libraries push their books at the same time and when they are ready.

The goal is to maximize the scanning score of all books within D days.

Your task Write down this book scanning problem as an integer program. State clearly your variable(s), cost function and constraint(s).

4.2 Solving a small book scanning problem (7 points)

Now suppose there are two libraries ($L = 2$).

There are 6 types of books ($B = 6$)

blue, red, green, yellow, purple and orange, labeled as $= i = \{1, 2, 3, 4, 5, 6\}$,

with the scores $\{1, 0.75, 1.5, 1, 0.5, 2\}$.

- Library A has $N_1 = 5$ books $L_1 = \{1, 2, 3, 4, 5\}$
- Library B has $N_2 = 4$ books $L_2 = \{1, 3, 4, 6\}$.
- Library A has a signup time $T_1 = 3$ days
- Library B has a signup time $T_2 = 2$ days.

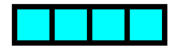
Both libraries can only ship one book per day. You are given $D = 6$ days. Write down this problem as an integer program, then solve this book scanning problem **WITHOUT** using MATLAB, and explain how you obtain the solution.

5 (Modified) Necklace splitting problem (20 points)

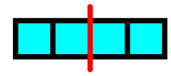
You and your friend have a necklace. You want to cut the necklace into 2 pieces, and each one of you get one piece.

To model the problem, consider the necklace as a long chain of diamond blocks, to avoid decreasing the value of the necklace, you should not cut at the block (cutting the diamond is not allowed) and you only cut the chain between the blocks.

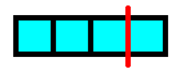
Example 1 Here is an example. You have a necklace with 4 blocks, and all block contains 1 diamond. All the diamonds are equal in value.



Splitting option 1 You decided to split it between the 2nd and the 3rd diamond.



Splitting option 2 You decided to split it between the 3rd and the 4th diamond.



Fair division For the splitting option 1, both of you get 2 diamonds, meanwhile for splitting option 2, one of you get fewer diamonds, the division is not fair so one of you are angry. To avoid conflict, you two agree to split the necklace **as fair as possible**.

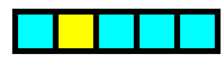
5.1 Splitting a 4-diamond necklace with one cut (5 points)

Given a necklace with 4 diamonds that all diamonds are equal valued, prove that the fairest solution to divide the necklace is to cut the necklace in the middle.

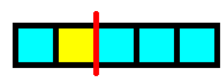
What you need to do Model the necklace splitting problem mentioned above as a linear integer program. Explain your variable(s), objective function and and constraint(s). Solve your model to confirm the solution.

5.2 Splitting an irregular necklace with one cut (6 points)

Example 2 You have another necklace. This time the necklace has 5 blocks, and it is made of diamond and gold.



Knowing that one gold has the same value as two diamonds, you two decided to split the necklace between the 2nd and the 3rd position. Each of you get a value equivalent to 3 diamonds. As the division is fair, both of you are happy.



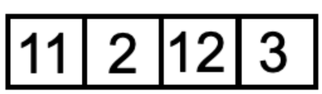
The general problem: irregular necklace Now, given the following

- A number n, referring to the number of blocks in the necklace
- A list of numbers $v_1, v_2, \dots, v_n$, referring to the value of the i^{th} block

Write down a linear integer program of deciding how to split the necklace (cut the necklace into two pieces) such that the division of the values is **as fair as possible**. Explain your variable(s), objective function and constraint(s).

5.3 Splitting an irregular necklace with two cuts (9 points)

Sometimes splitting the necklace into multiple pieces gives a division that is even more fair. For example, you two are given the following necklace.



The number on the block represents the value of that block. For this necklace, if only one cut is allowed, then fairest splitting of the necklace will be a cut between the 2nd and the 3rd blocks, which results in a split of values as $\{13, 15\}$. Although this is the best you can achieve in 1 cut, you two are not satisfied with such division.

Now you two agree to split the necklace into **3 pieces**. Write down a linear integer program of this necklace splitting problem. Explain how you derive the model, how you define the variable(s), what are the modelling constraint(s). Solve the model.

END of Final assignment.