Solution to CO327 (2022Spring) Midterm Assignment

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1 Coins Change Problem (7 points)

Problem description A country uses N coins with denominations $\{a_1, a_2, ..., a_N\}$. Given a value V, find the minimum number of coins that add up to V.

Example The coins in Japanese Yen are $\{1, 5, 10, 50, 100, 500\}$. Peter wants $V = 678$. The minimum number of coins required to make V is 9:

One 500 yen coin	Two 10 yen coin
One 100 yen coin	One 5 yen coin
One 50 yen coin	Three 1 yen coin

Your tasks

- Formulate the problem (NOT the example) as a linear program/integer program. State clearly your decision variable(s), objective function and constraint(s).
- Solve the problem for $N = 3$ with $a_1 = 0.5, a_2 = 1, a_3 = 2$ and $V = 5.5$.

Solution Part 1 Let $x_1, \dots, x_N$ be the amount of coin i to be taken. 1 (point)

The objective function is

$$\sum x_i \qquad \text{(1 point)}$$

explicit constraint 1: amount of money equal to V

$$\sum_i a_i x_i = V \qquad \text{(1 point)}$$

Implicit constraint 1: cannot pick negative amount

$$x_i \geq 0 \ \forall i \qquad \text{(1 point)}$$

Implicit constraint 2: pick integer amount of item

$$x_1 \in \mathbb{N} \ \forall i \qquad \text{(1 point)}$$

So the LP is

$$\begin{array}{rl} \min\limits_{x_i,\dots,x_N} & x_1 + x_2 + \cdots + x_N \\ \text{s.t.} & \sum_i a_i x_i = V \\ & x_i \in \mathbb{Z}_+. \end{array} \qquad \text{(1 point for the min)}$$

(6 points if the student directly write down the LP and it is all correct. Otherwise, deduct the points according to the missing portion.)

Solution Part 2

$$\min_{x_i,\dots,x_3} \quad x_1 + x_2 + x_3 \qquad\qquad \min_{x_i,\dots,x_3} \quad \mathbf{1}_3^\top \mathbf{x}$$
$$\text{s.t.} \quad 0.5x_1 + x_2 + 2x_3 = 5.5 \quad \Longleftrightarrow \quad \text{s.t.} \quad [0.5\ 1\ 2]\mathbf{x} = 5.5$$
$$x_i \in \mathbb{Z}_+. \qquad\qquad x_i \in \mathbb{Z}_+.$$

Solution: Two a_3, one a_2 and 1 a_1. 1 point

```
clear;close all;clc

c     = [1 1 1]';       % cost vector c for the min problem
Aeq   = [0.5 1 2];      % matrix A for the equality constraint
beq   = 5.5;            % vector b for the equality constraint
intid = [1 2 3];        % integer index
l     = [0 0 0]';       % lower bound vector

[x, fval] = intlinprog(c,intid,[],[],Aeq,beq, l);
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2 Employee scheduling (13 points)

You have a 24-hour supermarket. The supermarket has the following minimal requirements for cashiers:

Period	1	2	3	4	5	6
Time of the day (in 24-hour format)	3-7	7-11	11-15	15-19	19-23	23-3
Wages	110	100	100	107	107	110
Minimum no. of cashiers needed	2	10	4	12	4	2

Table 1: The wages and minimum number of cashiers for each work period.

Period 1 follows immediately after period 6. A cashier works 8 consecutive hours, starting at the beginning of one of the six periods. Different periods have different wages.

1. Your goal is to determine a daily employee worksheet (how many cashier in each time period) which satisfies the requirements with the least wages. Formulate this problem as a linear program/integer program. State clearly your decision variable(s), objective function and constraint(s). (8 points)
2. Solve the problem, how many cashier you need in each period? What is the total wages? (2 points)
3. At the solution, identify which constraint is active and which constraint is inactive. (3 points)

2.1 Solution

Let $x_1, \ldots, x_6$ be the number of cashier beginning work at the start of period i. (1 point, 1/6 each). The objective function is the sum of wages. The following objective function is wrong

$$110x_1 + 100x_2 + 100x_3 + 107x_4 + 107x_5 + 110x_6. \qquad \text{(WRONG)}$$

Why wrong: this means you only pay 4 hour wages to a staff working for 8 hours.

The correct one

$$(110+100)x_1 + (100+100)x_2 + (100+107)x_3 + (107+107)x_4 + (107+110)x_5 + (110+110)x_6. \quad \text{(1pt)}$$

There 6 explicit constraints

$$\begin{array}{lllllll} x_1 & & & & & +x_6 & \geq 2 \\ x_1 & +x_2 & & & & & \geq 10 \\ & x_2 & +x_3 & & & & \geq 4 \\ & & x_3 & +x_4 & & & \geq 12 \\ & & & x_4 & +x_5 & & \geq 4 \\ & & & & x_5 & +x_6 & \geq 2 \end{array} \qquad \text{(3 points, 3/6 each)}$$

Two implicit constraints: x_i are nonnegative (1 point) and integral (1 point). The LP is

$$\begin{array}{rl} \min\limits_{x_1,\ldots,x_6} & 210 + 200x_2 + 207x_3 + 214x_4 + 217x_5 + 220x_6 \\ \text{s.t.} & x_1 + x_6 \geq 2 \\ & x_1 + x_2 \geq 10 \\ & x_2 + x_3 \geq 4 \\ & x_3 + x_4 \geq 12 \\ & x_4 + x_5 \geq 4 \\ & x_5 + x_6 \geq 2 \\ & x_1, x_2, \ldots, x_6 \text{ are nonnegative and integral} \end{array} . \qquad \text{(1 point for the min)}$$

(8 points if the student directly write down the LP and it is all correct. Otherwise, deduct the points according to the missing portion.)

Correct solution

$$\mathbf{x}^* = \begin{bmatrix} 0 \\ 10 \\ 8 \\ 4 \\ 0 \\ 2 \end{bmatrix} \qquad \text{(1 pt.)}$$

You need to spend 4952$ for wages. 1 pt

At $\mathbf{x}^*$, constraints $\{1,2,4,5,6\}$ are active and $\{3\}$ is inactive. (3 pt, 0.5 each)
(Or $x_2 \geq 0, x3 \geq 0, x_4 \geq 0, x_6 \geq 0$ are inactive and $x_1 \geq 0, x_5 \geq 0$ are active)

Note: "x_i is integer" is always true but we do not need to consider is it active or not active.

3 Linear Programming (22 points)

Consider the following problem

$$\begin{array}{ll} \min & -2x_1 - x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 3 \\ & \left|x_1 + 4x_2\right| \leq 5 \\ & x_1 \geq 0 \\ & -x_2 \leq 0 \\ & |x_3| \geq 0 \end{array} \tag{1}$$

1. Convert (1) to canonical form
$$\begin{array}{ll} \max\limits_{\mathbf{x}} & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{array} \tag{1'}$$
2. Identify any redundant constraint in (1′).
3. Solve (1′) , give an optimal solution point $\mathbf{x}^*$.
4. State the active set S of $\mathbf{x}^*$. Based on $\mathbf{A}_S\mathbf{x}^* = \mathbf{b}_S$, what can you tell about the uniqueness of the solution to (1′)?
5. Convert (1′) to the following form
$$\begin{array}{ll} \max\limits_{\mathbf{x}} & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \tag{1''}$$
Then derive the symmetric dual of (1″) and solve it. Verify the duality and the complementary slackness between the primal and the dual problems.

Sol

1. Note that x_3 is redundant so it can be removed. 1 pt.
$$\begin{array}{ll} \max & \begin{bmatrix} 2 \\ 1 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{s.t.} & \begin{bmatrix} 2 & 1 \\ 1 & 4 \\ -1 & -4 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 5 \\ 5 \\ 0 \\ 0 \end{bmatrix} \end{array} \qquad \text{(3 pt.)}$$
2. $|x_3| \geq 0$ is redundant (if haven't remove it in part 1) 1 pt.
$-1x_1 - 4x_2 \leq 5$ is redundant (because of $x_1 \geq 0, x_2 \geq 0$) 1 pt.
3. The solution is $x^* = [1.5, 0]^\top$. 1 pt.
4. (If x_3 is removed) The active set of $\mathbf{x}^*$ is $\{1, 5\}$ 2 pt.
(If x_3 is not removed, add back the row index correspond to x_3.)
The solution is non-unique 1 pt.
(In fact, any point on $2x_1 + x_2 \leq 3$ is also a sol because $\mathbf{c}$ is exactly the normal to such hyperplane. The $\mathbf{A}_S\mathbf{x}^* = \mathbf{b}_S$ argument here is useless and it is used to confuse student.)

5. The new problem (after redundant constraints removed)

$$\begin{aligned} \max \quad & \begin{bmatrix} 2 \\ 1 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \qquad \text{(4 pt.)}$$

The symmetric dual

$$\begin{aligned} \min \quad & \begin{bmatrix} 3 \\ 5 \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned} \qquad \text{(4 pt.)}$$

Sol: $\mathbf{y}* = [1, 0]^\top$. 1 pt.
Strong duality is true: $\mathbf{c}^\top \mathbf{x}^* = 3 = \mathbf{b}^\top \mathbf{y}^*$. 1 pt.

Complementary slackness : yes by checking $\mathbf{y}^* \geq \mathbf{0}$ vs $\mathbf{A}\mathbf{x}^* - \mathbf{b}$ and $\mathbf{x}^* \geq \mathbf{0}$ vs $\mathbf{A}^\top \mathbf{y}^* - \mathbf{c}$ 2 pt.
(Note that it might seem not true at first glance)

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