CO327 Deterministic OR Models (2022-Spring) Pari-mutuel auction

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"Traditional" trading vs auction



- Traditional trading
 - ► A "seller" and a (single) "buyer"
 - Seller determines the price
 - Buyer, knows the price (from himself), determines the amount of goods and thereby how much to pay
- Auction
 - ► An "auction organizer" and a (bunch of) "bidder(s)"
 - Bidder proposes the price
 - Organizer, knows the price (from bidder), determines the amount of goods and thereby how much to sell
 - Price determination mechanism

- Terminology of auction
 - State, state prices
 - Order $[a, \pi, q]$, from bidders
 - State indicator a
 - Limit: Price limit π and Quantity limit q
 - Share x, to be determine by auction organizer
- ► Example: 2018 FIFA World Cup betting
 - ▶ 4 teams have a chance to win: France, Belgium, Croatia, England
 - ► What's the market: bid which team win.
 - Motivation of auction: to tell "how much it cost for betting France win?"
 - Seller don't know about the price determination via auction).

Some historical background of auction

- ► Auction: long history, google it yourself.
- 2020 Nobel Prize for Economics was on auction theory: "for improvements to auction theory and inventions of new auction formats"
- ► What we will look at: Pari-mutuel auction
 - ▶ was developed in 1864.
 - the mathematics started around 2000: Bossaerts, P., L. Fine, J. Ledyard: "Inducing Liquidity in Thin Financial Markets Through Combined-Value Trading Mechanisms", European Economic Review, 2002
 - What can you do with the knowledge auction: bet horse racing.

Don't gamble.



Auction terminology: state

- ▶ (2018 World cup) 4 teams: France, Belgium, Croatia, England
- ► There are 4 possible *states* (outcomes)
 - ► State 1: France win
 - State 2: Belgium win
 - State 3: Croatia win
 - State 4: England win
- Properties of the states
 - Mutually exclusive.
 - e.g. Impossible for both France and Belgium to win.
 - Exactly one will be realized / will happen Impossible for no winner / all lose.

What bidders and organizer do in auction

- Each bidder proposes an *order* (offer): $[a, \pi, q]$
 - 1. a : indication vector of "biding which team will win". This is the "choice/decision" chosen by the bidder
 - 2. How much the bidder is willing to pay
 - π : Maximum *limit price* per bid
 - q : Maximum *limit of quantity* of bid

They represent the maximum amount of "risk", or the "budget" the bidder can bear

- Organizer determines the "share" (order fill) x of each bidder.
 x = the number of "lottery ticket" sold to each bidder
- A "contract" = if the order include the winning state, it worth 1 (normalized), 0\$ else.
 - Example: Bidder 1 has 5 tickets bidding on France, Bidder 2 has 6 tickets bidding on England. In the end France won, so bidder 1 won 5\$, bidder 2 won 0\$

Auction terminology: Order $[\boldsymbol{a}, \pi, q]$

- \blacktriangleright Come from bidders $1,2,\ldots,n$
- ► The (row) order vector

$$i^{\mathsf{th}} \; \mathsf{order} = \left[oldsymbol{a}_{i:} \in \{0,1\}^m, \; \pi_i \in \mathbb{R}_+, \; q_i \in \mathbb{R}_+
ight]$$

- $a_{i:}$: a vector of 0-1 indicating the bid
 - e.g. "France win" = [1, 0, 0, 0]
 - e.g. "England win" = [0, 0, 0, 1]
 - e.g. "either France or England win" = [1, 0, 0, 1]
- π_i and q_i : how much the bidder is willing the pay
 - π_i : price limit per share
 - q_i : max share
 - πq_i = budget upper bound / maximum bearable risk of bidder i
- The *n* vectors a_1, a_2, \ldots, a_n collectively formed a matrix A.

Money transfer

- Recall x_i is called the share (amount of order of bidder i),
 - ▶ it represents the number of tickets for bidder *i*.
 - ▶ it is to be determined by the organizer by a "mechanism".
- Bidder *i* pays $\pi_i x_i$ money to the organizer.
- ► The organizer has the *total grand* sum / *pool* of the bidding money

$$\sum_{i=1}^{n} \pi_i x_i$$

 \blacktriangleright Winner of the bid get 1\$ per share of contract, the organizer pays all the winners

$$\sum_{i \in \text{winners}} 1\$ \cdot x_i = \sum_{i \in \text{winners}} x_i$$

Pari-mutuel auction market

- ► A word with historical context (google it yourself)
- ► What it means: *Winners take all*
 - Winners = the order vector a_i: containing the winning state. i.e. those who guess correctly the team winning world cup.
 - Take = divide the money in proportional to how they bet individually. You get 1\$ per share, you get more if you have more share x
 - ► All = grand pool <u>management expense</u> let's assume = 0

A concrete example: simple setup

- ► Bidding item: will Belgium win?
 - Either yes or no: binary choice
 - Order indication vector $\boldsymbol{a} = [*,*] \in \{0,1\}^2$
- ▶ 3 bidders: Amy, Bob, Peter and their order

Bidder	a	π	q
Amy	[1,0]	0.75	10
Bob	[0,1]	0.35	5
Peter	[1,1]	0.4	10

The table means

- Amy set her price limit per share as 0.75^{\$}: she is willing to pay at most 0.75^{\$} per share.
- ▶ Bob set his quantity limit as 5: he is willing to buy at most 5 unit of the order.
- ▶ Peter bids Belgium win and Belgium lose: he will always be a winner.

A concrete example: the task of the organizer

 \blacktriangleright The task of the auction organizer is to fill in the x values

Bidder	a	π	q	x
Amy	[1,0]	0.75	10	x_{amy}
Bob	[0,1]	0.35	5	x_{bob}
Peter	[1,1]	0.4	10	x_{peter}

subject to constraints

$$0 \le x_{\text{amy}} \le 10, \quad 0 \le x_{\text{bob}} \le 5, \quad 0 \le x_{\text{peter}} \le 10$$

▶ The organizer fill the values of x that maximizes an objective: profit

A concrete example: profit of the organizer

Bidder	\boldsymbol{a}	π	\boldsymbol{q}	\boldsymbol{x}
Amy	[1,0]	0.75	10	x_{amy}
Bob	[0,1]	0.35	5	x_{bob}
Peter	[1, 1]	0.4	10	x_{peter}

Auction organizer collect

$$\mathsf{pool} = 0.75x_{\mathsf{amy}} + 0.35x_{\mathsf{bob}} + 0.4x_{\mathsf{peter}} = \boldsymbol{\pi}^{\top}\boldsymbol{x}$$

- Winning money = pool management expense. Suppose 0 management expense, the winner(s) get money from the pool among themselves proportionally.
- ► Suppose Belgium lose, so winners are Bob and Peter.
 - Recall: contract = if the order includes the winning state, it worths 1 (normalized), 0\$ else.
 - Auction organizer has to pay 1\$ per each winning contract, the organizer has to pay: 1 - m = -1 - m = -1 - m
 - $1 \cdot x_{bob} + 1 \cdot x_{peter} = x_{bob} + x_{peter}$
 - Profit of the organizer = pool $(x_{bob} + x_{peter})$

A concrete example: worst-case profit

	Bidder	\boldsymbol{a}	π	${m q}$	\boldsymbol{x}	Г1	٥٦
	Amy	[1,0]	0.75	10	x_{amy}	$\mathbf{A} = \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0\\1 \end{bmatrix} = \begin{bmatrix} a \\ 1 \end{bmatrix} \begin{bmatrix} a \\ 2 \end{bmatrix}$
	Bob	[0,1]	0.35	5	x_{bob}	$\mathbf{A} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ - \begin{bmatrix} \boldsymbol{u}_{:,1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_{:,2} \end{bmatrix}$
	Peter	[1, 1]	0.4	10	x_{peter}	L	1
Note that a	$x_{bob} + x_{pe}$	ter =	$\begin{bmatrix} 0\\1\\1 \end{bmatrix}^{\top} \begin{bmatrix} z\\z \end{bmatrix}$	$x_{amy} \\ x_{bob} \\ x_{peter}$	$= \boldsymbol{a}_{:2}^{ op} \boldsymbol{x}.$		

- Net gain of organizer (if Belgium lose) = $\pi^{\top} x a_{:2}^{\top} x$.
- In general organizer doesn't know which state will win, so it is natural to consider maximizing the worst-case profit:

$$\max_{oldsymbol{x}} \left\{ oldsymbol{\pi}^ op oldsymbol{x} - \max_j oldsymbol{a}_{:j}^ op oldsymbol{x}
ight\}$$

subject to constraints: $0 \le x_i \le q_i$.

►

- ► Amy buys at most 10 contracts, at least 0 contract.
- ▶ Bob buys at most 5 contracts, at least 0 contract.
- \blacktriangleright Peter buys at most 10 contracts, at least 0 contract

A concrete example: worst-case profit – explaining $\max_{j \in \{1,2\}} \left\{ \boldsymbol{a}_{:j}^{ op} \boldsymbol{x} \right\}$

Bidder	\boldsymbol{a}	π	\boldsymbol{q}	\boldsymbol{x}
Amy	[1,0]	0.75	10	x_{amy}
Bob	[0,1]	0.35	5	x_{bob}
Peter	[1, 1]	0.4	10	x_{peter}

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = [\boldsymbol{a}_{:,1} \mid \boldsymbol{a}_{:,2}].$$

- ► If Belgium win
 - Bob and Peter are the winners.
 - Organizer has to pay them $a_{:2}^{ op}x$ \$ in total
- ► If Belgium lose
 - ► Amy and Peter are the winners.
 - Organizer has to pay them $a_{:1}^ op x$ in total
- The worst of the two

$$\max\left\{\boldsymbol{a}_{:2}^{\top}\boldsymbol{x}\,,\,\boldsymbol{a}_{:1}^{\top}\boldsymbol{x}\right\}\ =\ \max_{j\in\{1,2\}}\left\{\boldsymbol{a}_{:j}^{\top}\boldsymbol{x}\right\}$$

A concrete example: the Pari-mutuel auction model

Bidder	a	π	q	x
1	[1,0]	0.75	10	x_1
2	[0,1]	0.35	5	x_2
3	[1,1]	0.4	10	x_3

For the example

$$\max_{x} \quad 0.75x_1 + 0.35x_2 + 0.4x_3 - \max\{x_2 + x_3, x_1 + x_3\}$$

s.t.
$$0 \le x_1 \le 10$$
$$0 \le x_2 \le 5$$
$$0 \le x_3 \le 10$$

In general form:

$$\max_{\boldsymbol{x}} \quad \left\{ \boldsymbol{\pi}^{\top} \boldsymbol{x} - \max_{j} \boldsymbol{a}_{:j}^{\top} \boldsymbol{x} \right\}$$
 s.t. $\boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{q}$

Pari-mutuel auction is a conic optimization problem

$$egin{array}{cc} \max_{m{x}} & m{\pi}^{ op} m{x} - \max_{j} m{a}_{:j}^{ op} m{x} \ ext{s.t.} & m{0} \leq m{x} \leq m{q} \end{array}$$

- ► This is a linear conic programming.
- Conic constraint: $x \in \mathcal{K}$, where \mathcal{K} is a cone.

$$oldsymbol{u},oldsymbol{v}\in\mathcal{K}\implies aoldsymbol{u}+boldsymbol{v}\in\mathcal{K}$$
 for any $a,b\geq 0$

▶ Using ℓ_{∞} norm notation

$$egin{array}{lll} \max_{m{x}} & \pi^{ op} m{x} - \|m{A}^{ op} m{x}\|_{\infty} \ {
m s.t.} & m{0} \leq m{x} \leq m{q} \end{array}$$

LP form of Pari-mutuel auction

$$\begin{array}{ll} \max_{\boldsymbol{x}} & \boldsymbol{\pi}^{\top}\boldsymbol{x} - \max_{j} \boldsymbol{a}_{:j}^{\top}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{q} \end{array} \tag{Not a LP}$$

Using the worst-case modeling trick

$$\begin{array}{ll} \max_{\boldsymbol{x},y} & \boldsymbol{\pi}^{\top}\boldsymbol{x} - y \\ \text{s.t.} & \boldsymbol{A}^{\top}\boldsymbol{x} \leq \mathbf{1}y \\ & \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{q} \end{array} \tag{A LP}$$

- ▶ How to solve: turn this into standard form / canonical form, call solver to solve it.
- Theory: the optimal solution to the dual problem of the Pari-mutuel auction is the price index for each team.

Why the dual optimal solution is the price

$$\begin{array}{ll} \max_{\boldsymbol{x}} & \boldsymbol{\pi}^\top \boldsymbol{x} - \max_{j} \boldsymbol{a}_{:j}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{q} \end{array}$$

- ▶ x is the share, it represents the number of "tickets"
- ► Recall the strong duality

$$\pi^ op x = q^ op y$$

- π is the price limit so it carries the unit of \$
- $\pi^{\top} x$ carries the unit of \$ as x is quantity (no unit)
- $\pi^{ op} x = q^{ op} y$ means $q^{ op} y$ carries the unit of \$
- q has no unit so y has the unit of \$

Back to the concrete example

Using the worst-case modeling trick, introduce a variable t

$$\max_{\substack{x,y \\ \text{s.t.}}} \quad 0.75x_1 + 0.35x_2 + 0.4x_3 - t \\ x_2 + x_3 \le t \\ x_1 + x_3 \le t \\ 0 \le x_1 \le 10, \quad 0 \le x_2 \le 5, \quad 0 \le x_3 \le 10$$

Solving this gives the optimal share distribution.



- ► Introduction to auction
- ► Pari-mutuel auction
- ► Formulation of Pari-mutuel auction

End of document