CO327 Deterministic OR Models (2021-Spring)

Pari-mutuel auction and introduction to conic programming

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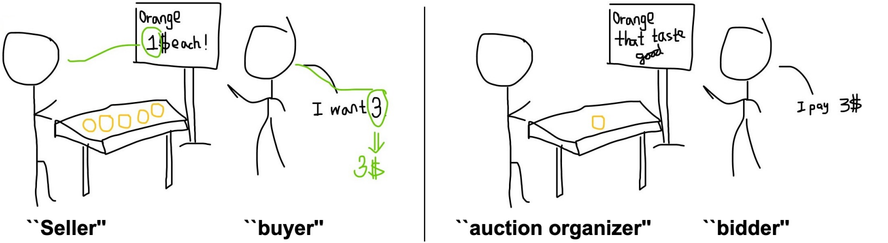
"Traditional" trading vs auction



- Traditional trading
  - A "seller" and a (single) "buyer"
  - Seller determines the **price**
  - Buyer, **knows the price**, determines the **amount** of goods and thereby **how much to pay**
- Auction
  - An "auction organizer" and a (bunch of) "bidder(s)"
  - Bidder proposes the **price**
  - Organizer, **knows the price**, determines the **amount** of goods and thereby **how much to sell**
  - Price determination mechanism

Terminology and 2018 FIFA World Cup betting

- Terminology
  - State, state prices
  - Order $[\mathbf{a}, \pi, q]$
    - State indicator $\mathbf{a}$
    - Limit: Price limit π and Quantity limit q
  - Share x, to be determine by auction organizer
- 2018 FIFA World Cup betting
  - 4 teams have a chance to win: France, Belgium, Croatia, England
  - What's the market: bid which team win.
  - Motivation of auction: how to tell "how much it cost for betting France win?"
  - Seller don't know about the price $\implies$ not traditional trading but auction (price determination).

About auction

- Auction: long history, google it yourself.
- 2020 Nobel Prize for Economics was on auction theory.
- What we will look at: Pari-mutuel auction mechanism.
- Mathematics of Pari-mutuel auction mechanism: started around 2000.
- What can you do with the knowledge auction: bet horse racing.

Don't gamble.

Auction terminology: state

- (2018 World cup) 4 teams: France, Belgium, Croatia, England

- There are 4 possible *states* (outcomes)
  - State 1: France win
  - State 2: Belgium win
  - State 3: Croatia win
  - State 4: England win

- Properties of the states
  - Mutually exclusive.
    e.g. Impossible for both France and Belgium to win.

  - Exactly one will be realized / will happen
    Impossible for no winner / all lose.

What bidders and organizer do in an auction

- Bidder proposes *order* (offer)
  1. $\mathbf{a}$: indication vector of "biding which team will win".
     The "choice/decision" chosen by the bidder

  2. How much the bidder is willing to pay
     - π: Maximum *limit price* per bid
     - q : Maximum *limit of quantity* of bid

     The maximum amount of "risk" the bidder can bear"

- Organizer determines the "*share*" (order fill) x of each bidder.

- Contract = if the order include the winning state, it worth 1$ (normalized) , 0$ else.

Auction terminology: Order

- Come from bidders

- The (row) order vector

$$i^{\text{th}} \text{ order} = \left[\mathbf{a}_{i:} \in \{0,1\}^m,\ \pi_i \in \mathbb{R}_+,\ q_i \in \mathbb{R}_+\right]$$

  - $\mathbf{a}_{i:}$: the bid indication
    - e.g. "I bid France win" $= [1, 0, 0, 0]$
    - e.g. "I bid England win" $= [0, 0, 0, 1]$
    - e.g. "I bid either France or England win" $= [1, 0, 0, 1]$

    $\mathbf{a}_{i:}$ is a vector of 0-1

  - π_i and q_i: how much the bidder is willing the pay
    - π_i: price limit per share
    - q_i: max share

- There are n bidders, so we have n vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, collectively as a matrix $\mathbf{A}$.

Money transfer

- The ith bidder pays $\pi_i x_i$ money to the organizer.
- The organizer collects $\sum_{\text{all bidder}} \pi_i x_i$ money,
  in other words, the *total grand* sum / *pool* of the bidding money is
  $$\sum_{i=1}^{n} \pi_i x_i$$
- x_i: the share / amount of order of the ith bidder, to be determined by the organizer by a "mechanism".
- Winner of the bid get 1\$ per share of contract, the organizer pays all the winner
  $$\sum_{i \in \text{winners}} 1\$ \cdot x_i = \sum_{i \in \text{winners}} x_i$$

Pari-mutuel auction market mechanism

- A word with historical context (google it yourself)
- What it means: *Winners take all*
  - Winners $=$ the order vector $\mathbf{a}_{i:}$ containing the winning state.
    i.e. those who guess correctly the team winning world cup.
  - Take $=$ divide the money in proportional to how they bet individually.
    You get 1\$ per share, you get more if you have more share x
  - All $=$ grand pool - $\underbrace{\text{management expense}}_{\text{let's assume } = 0}$

A concrete example: simple setup

- Bidding item: will Belgium win?
  - Either yes or no: binary choice
  - Order indication vector $\mathbf{a} = [*, *] \in \{0, 1\}^2$
- 3 bidders: Amy, Bob, Peter and their order

Bidder	$\mathbf{a}$	π	q
Amy	[1,0]	0.75	10
Bob	[0,1]	0.35	5
Peter	[1,1]	0.4	10

- Amy set her price limit per share as $0.75\$$: she is willing to pay at most $0.75\$$ per share.
- Bob set his quantity limit as 5: he is willing to buy at most 5 unit of the order.
- Peter bids Belgium win and Belgium lose: he will always be a winner.

A concrete example: the task of the organizer

- The task of the auction organizer is to fill in the following table

Bidder	$\mathbf{a}$	π	q	x
Amy	[1,0]	0.75	10	x_{amy}
Bob	[0,1]	0.35	5	x_{bob}
Peter	[1,1]	0.4	10	x_{peter}

  subject to constraints

$$0 \leq x_{\text{amy}} \leq 10, \quad 0 \leq x_{\text{bob}} \leq 5, \quad 0 \leq x_{\text{peter}} \leq 10$$

- The organizer find the values of x that maximize a objective: profit

A concrete example: profit of the organizer

Bidder	$\mathbf{a}$	$\boldsymbol{\pi}$	$\mathbf{q}$	$\mathbf{x}$
Amy	[1,0]	0.75	10	x_{amy}
Bob	[0,1]	0.35	5	x_{bob}
Peter	[1,1]	0.4	10	x_{peter}

- Auction organizer collect

$$\text{pool} = 0.75x_{\text{amy}} + 0.35x_{\text{bob}} + 0.4x_{\text{peter}} = \boldsymbol{\pi}^\top \mathbf{x}$$

- Winning money = pool - management expense. Suppose 0 management expense, the winner(s) get money from the pool among themselves proportionally.
- Suppose Belgium lose, so winners are Bob and Peter.
  - Recall: contract = if the order include the winning state, it worth 1$ (normalized) , 0$ else.
  - Auction organizer need to pay 1$ per each winning contract, the organizer has to pay: $1 \cdot x_{\text{bob}} + 1 \cdot x_{\text{peter}}$
  - Profit of the organizer = pool - $(x_{\text{bob}} + x_{\text{peter}})$

A concrete example: worst-case profit

Bidder	$\mathbf{a}$	$\boldsymbol{\pi}$	$\mathbf{q}$	$\mathbf{x}$
Amy	[1,0]	0.75	10	x_{amy}
Bob	[0,1]	0.35	5	x_{bob}
Peter	[1,1]	0.4	10	x_{peter}

- Note that $x_{\text{bob}} + x_{\text{peter}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}^\top \begin{bmatrix} x_{\text{amy}} \\ x_{\text{bob}} \\ x_{\text{peter}} \end{bmatrix} = \mathbf{a}_{:2}^\top \mathbf{x}$.
- Net gain of organizer (if Belgium lose)$= \boldsymbol{\pi}^\top \mathbf{x} - \mathbf{a}_{:2}^\top \mathbf{x}$.
  But in general organizer doesn't know which state will win.
- Maximizing the worst-case profit:

$$\max_{\mathbf{x}} \left\{ \boldsymbol{\pi}^\top \mathbf{x} - \max_j \mathbf{a}_{:j}^\top \mathbf{x} \right\}$$

  subject to constraints: $0 \leq x_i \leq q_i$.
  - Amy buys at most 10 contracts, at least 0 contract.
  - Bob buys at most 5 contracts, at least 0 contract.
  - Peter buys at most 10 contracts, at least 0 contract

A concrete example: the Pari-mutuel auction

Bidder	$\mathbf{a}$	π	q	x
1	[1,0]	0.75	10	x_1
2	[0,1]	0.35	5	x_2
3	[1,1]	0.4	10	x_3

Example-specific:

$$\begin{aligned} \max_{\mathbf{x}} \quad & 0.75x_1 + 0.35x_2 + 0.4x_3 - \max\{x_2, x_3\} \\ \text{s.t.} \quad & 0 \leq x_1 \leq 10 \\ & 0 \leq x_2 \leq 5 \\ & 0 \leq x_3 \leq 10 \end{aligned}$$

General form:

$$\begin{aligned} \max_{\mathbf{x}} \quad & \left\{ \boldsymbol{\pi}^\top \mathbf{x} - \max_j \mathbf{a}_{:j}^\top \mathbf{x} \right\} \\ \text{s.t.} \quad & \mathbf{0} \leq \mathbf{x} \leq \mathbf{q} \end{aligned}$$

Pari-mutuel auction

$$\begin{aligned} \max_{\mathbf{x}} \quad & \boldsymbol{\pi}^\top \mathbf{x} - \max_j \mathbf{a}_{:j}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{0} \leq \mathbf{x} \leq \mathbf{q} \end{aligned}$$

- This is a linear conic programming.
- Conic constraint: $\mathbf{x} \in \mathcal{K}$, where $\mathcal{K}$ is a cone.

$$\mathbf{u}, \mathbf{v} \in \mathcal{K} \implies a\mathbf{u} + b\mathbf{v} \in \mathcal{K} \text{ for any } a, b \geq 0$$

- Using ℓ_∞ norm notation

$$\begin{aligned} \max_{\mathbf{x}} \quad & \boldsymbol{\pi}^\top \mathbf{x} - \|\mathbf{A}^\top \mathbf{x}\|_\infty \\ \text{s.t.} \quad & \mathbf{0} \leq \mathbf{x} \leq \mathbf{q} \end{aligned}$$

LP form of Pari-mutuel auction

$$\begin{aligned} \max_{\mathbf{x}} \quad & \boldsymbol{\pi}^\top \mathbf{x} - \max_j \mathbf{a}_{:j}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{0} \le \mathbf{x} \le \mathbf{q} \end{aligned}$$

- Using the worst-case modeling trick

$$\begin{aligned} \max_{\mathbf{x},y} \quad & \boldsymbol{\pi}^\top \mathbf{x} - y \\ \text{s.t.} \quad & \mathbf{A}^\top \mathbf{x} \le \mathbf{1} y \\ & \mathbf{0} \le \mathbf{x} \le \mathbf{q} \end{aligned}$$

- How to solve: turn this into standard form / canonical form, call solver to solve it.

- Theory (later in the theory lecture): the optimal solution to the dual problem of the Pari-mutuel auction is the price index for each team.

A concrete example

Bidder	$\mathbf{a}$	π	q	x
1	[1,0]	0.75	10	x_1
2	[0,1]	0.35	5	x_2
3	[1,1]	0.4	10	x_3

$$\begin{aligned} \max_{\mathbf{x}} \quad & 0.75x_1 + 0.35x_2 + 0.4x_3 - \max\{x_2, x_3\} \\ \text{s.t.} \quad & 0 \le x_1 \le 10, \quad 0 \le x_2 \le 5, \quad 0 \le x_3 \le 10 \end{aligned}$$

Introducing the worst-case variable y

$$\begin{aligned} \max_{\mathbf{x}, y} \quad & 0.75x_1 + 0.35x_2 + 0.4x_3 - y \\ \text{s.t.} \quad & x_2 \le y \\ & x_3 \le y \\ & 0 \le x_1 \le 10, \quad 0 \le x_2 \le 5, \quad 0 \le x_3 \le 10 \end{aligned}$$

Assignment 3: turn this into standard form and canonical form.

Summary

- Introduction to auction
- Pari-mutuel auction
- Formulation of Pari-mutuel auction

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