CO327 (2021Spring) Final Assignment

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Instructions

- This is a take-home final exam and this counts 25% of the course grade.
- In your PDF, write down your name and your student ID.
- You have to generate your PDF using LaTeX. However, for figures, you can generate it by any method you like. (To include graphics in LaTeX, see `https://www.overleaf.com/learn/latex/Inserting_Images`). **You will get -10% if you do not generate the PDF using LaTeX**.
- Use of MATLAB is allowed. If you use MATLAB to solve a problem, **you do NOT need to provide the code and I will not read your code**. Write down the numerical solution directly in the PDF.
- There are six questions. Answer ALL the questions.
- Try your best in answering the questions. Some questions in this final are similar to those discussed during class. However, some are "hard" due to two reasons:
  1. some questions are hard because they are tricky.
  2. some questions are seemingly hard because they are designed to test your mathematical skills in facing unfamiliar cases.

  Try your best to solve them.
- Name your PDF as "Your student ID" + "final".pdf and submit to the dropbox in Waterloo LEARN.
- Submission deadline
  - August-16 23:55 (EDT), late submission get zero point.
  - You get bonus points if
    * Submit it before August-6 23:55 (EDT), + 5%
    * Submit it before August-8 23:55 (EDT), + 4%
    * Submit it before August-10 23:55 (EDT), + 3%
    * Submit it before August-12 23:55 (EDT), + 2%
    * Submit it before August-14 23:55 (EDT), + 1%
  - If you re-submit: everything will be based on your newest submission, and your previous submission file will be ignored.

1 Linear Programming (30 points)

Consider the following problem

$$\begin{array}{ll} \min & -2x_1 - x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 3 \\ & \left|x_1 + 4x_2\right| \leq 5 \\ & x_1 \geq 0 \\ & -x_2 \leq 0 \\ & |x_3| \geq 0 \end{array} \tag{1}$$

1. Convert (1) to canonical form

$$\begin{array}{ll} \max\limits_{\mathbf{x}} & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{array} \tag{1'}$$

2. Identify any redundant constraint in $(1')$.

3. Solve $(1')$, give an optimal solution point $\mathbf{x}^*$.

4. State the active set S of $\mathbf{x}^*$. Based on $\mathbf{A}_S\mathbf{x}^* = \mathbf{b}_S$, what can you tell about the uniqueness of the solution to $(1')$?

5. Convert $(1')$ to the following form

$$\begin{array}{ll} \max\limits_{\mathbf{x}} & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \tag{1''}$$

   Then derive the symmetric dual of $(1'')$ and solve it. Verify the duality and the complementary slackness between the primal and the dual problems.

6. Now consider the following ILP problem denoted as $\mathcal{P}(\mathbf{c}, \mathbf{A}, \mathbf{b})$:

$$\mathcal{P}(\mathbf{c}, \mathbf{A}, \mathbf{b}) : \left\{ \begin{array}{ll} \min & -2x_1 - x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 3 \\ & x_1 + 4x_2 \leq 5 \\ & x_i \in \{0, 1\}, \text{for all } i \end{array} \right.$$

   - Show that $\mathbf{A}$ is not unimodular.
   - Solve $\mathcal{P}(\mathbf{c}, \mathbf{A}, \mathbf{b})$ and confirm that LP relaxation works for this IP.
   - A student Alice said that "For $\mathcal{P}(\mathbf{c}, \mathbf{A}, \mathbf{b})$, the matrix $\mathbf{A}$ is not unimodular, however LP relaxation works for this problem, hence this contradicts to the lectures on the unimodularity, and therefore unidmoularity is wrong". Comment on her statement. Explain why even $\mathbf{A}$ is not unimodular, LP relaxation still works for $\mathcal{P}(\mathbf{c}, \mathbf{A}, \mathbf{b})$, and explain when will LP relaxation fail for $\mathcal{P}$.

7. Make it as simple as you can, make up a story of a LP problem that, when formulating such a problem will give raise to $(1'')$.

2 ℓ_1 and ℓ_∞ norms (15 points)

Consider the following problem

min ~~max~~
correction 2021-Aug-8 8:40pm

$$\begin{array}{ll} \min & \|\mathbf{Ax}-\mathbf{b}\|_\infty + 2\|\mathbf{x}\|_1 \\ \text{s.t.} & \mathbf{0} \le \mathbf{x} \le \begin{bmatrix} 5 \\ 4 \end{bmatrix} \end{array}, \quad \mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ p \end{bmatrix} \tag{2}$$

Solve (2) with $p = 2$, 20, 200, 2000. What did you observe? Explain what just happened.

3 Applications (20 points)

3.1 A story from One Thousand and One Nights/Arabia's night.

A king met 3 boys and told them about his mule. "What color is the mule?" asked one boy. "Either brown, black, or gray. Make a guess." answered the king. The boys made the following guesses:

Boy	1	2	3
Guess	Black	Either brown or gray	Brown

The king replied: "at least one of you guessed right, and at least one of you guessed wrong".

1. Formulate this problem as BIP.
2. Solve the BIP. Which boy guessed it right? What is the mule's color?

3.2 2-player zero-sum game

Two players: player 1 (P1) and player 2 (P2) play the Paper scissor rock game. (If you don't know this game, see `https://en.wikipedia.org/wiki/Rock_paper_scissors`.) From the perspective of P1, we have the following payoff table, where +1 means P1 is the winner, -1 means P1 loses and 0 means there is a draw.

	P2 Paper	P2 Scissor	P2 Rock
P1 Paper	0	- 1	+1
P1 Scissor	+1	0	-1
P1 Rock	-1	+1	0

Table 1: The payoff matrix of player 1 in the Paper Scissor rock game.

Now suppose P1 and P2 both use a probabilistic strategy to play this game: P1 will choose paper, scissor and rock with the probability x_P, x_S, x_R, respectively, and P2 will choose paper, scissor and rock with the probability y_P, y_S, y_R, respectively. Consider the situation that the two players will play the same game many many times. Use the theory of 2-player zero-sum game, prove that in this situation, the best strategy for P1 is to choose paper, scissor and rock with equal probability. In other words, prove that randomly selecting paper, scissor and rock with equal probability is the best way to play this game in the long run. (Hint: if you use MATLAB to solve the LP problem, use `linprog(c,A,b,Aeq,beq,l)`)

4 Production economics and labour allocation (15 points)

In this problem you will solve a simplified model of production economics proposed by a Nobel Prize laureate. You are the president of Canada. Canada's economy encompasses a network of interdependent industries.

We now explain the terms. **Good**: Each industry both produces and consumes "goods". E.g., the steel industry consumes coal to manufacture steel. **Industry**: Each industry requires different resources per unit production. E.g., one industry for producing steel starts with iron ore while another makes use of scrap metal. **Labour**: Canada is rich in natural resources so you can assume labour is the only limiting factor. Your task is to develop a model to decide how the labour force should be allocated among industries.

Production matrix: Each industry produces a single good and may consume others. There are M goods, indexed $i = 1, \ldots, M$. Each can be produced by one or more industries. There are a total of $N \geq M$ industries, indexed $j = 1, \ldots, N$. Each jth industry produces $A_{ij} > 0$ units of some ith good per unit of labour. For each $k \neq i$, this jth industry may consume some amount of good k per unit labour, denoted by $A_{kj} \leq 0$. Note that A_{kj} is nonpositive; if it is a negative number, it represents the quantity of good k consumed per unit labour allocated to industry j. The productivity and resource requirements of all industries are therefore captured by a matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$ in which each column has exactly one positive entry and each row has at least one positive entry. This matrix $\mathbf{A}$ is called the production matrix.

Modelling the constraints Suppose we have a total of one unit of labour to allocate over the next year. Let $\mathbf{x} \in \mathbb{R}^N$ be our allocation among the N industries, where entries of $\mathbf{x}$ refer to the fraction of labour allocated to each industry.

Q1. Write down the constraint associated with x.

Q2. Using A and x, Write down the quantity of each of the M goods produced.

The objective function The objective is be to optimize the social welfare. Suppose the amount society values each unit of each ith good is $c_i > 0$, regardless of the quantity produced. Suppose our objective function is the social welfare generated linearly by production of goods of all industries.

Q3. Write down the objective function in terms of $\mathbf{c}, \mathbf{A}$ and $\mathbf{x}$.

Q4. Write down the whole optimization problem.

Q5. An example Solve the labour force allocation problem for the following table. There are 3 type of goods and 3 type of industries. How should you distribute the labour force? What is the final amount of production for each type of goods? What is the final total social welfare?

Industry		1	2	3	Society value
	1	0.4	-0.1	0	1
Goods	2	-0.1	0.8	-0.1	3
	3	-0.1	-0.5	0.5	2

5 Duality of LP (25 points)

Notation In this question,

- $\mathbf{x}^*$ denotes the optimal solution of the primal problem
- $p^* = \mathbf{c}^\top \mathbf{x}^*$ denotes the optimal primal cost value / optimal objective function value of the primal problem
- $\mathbf{y}^*$ denotes the optimal solution of the dual problem
- $d^* = \mathbf{b}^\top \mathbf{y}^*$ denotes the optimal dual cost value / optimal objective function value of the dual problem

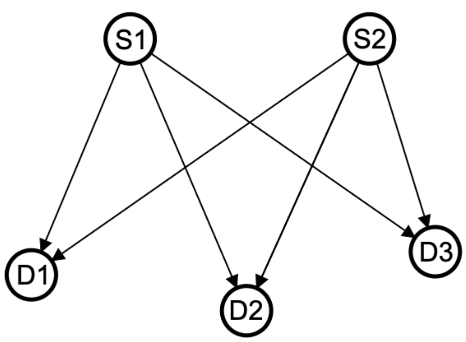
The problem Consider the following LP of single scalar variable: maximize cx subject to $ax \leq b$ and $x \geq 0$ with $a = 0, c = b = 1$. That is,

$$(P)\begin{cases} \max\limits_x & 1 \cdot x \\ \text{s.t.} & 0 \cdot x \leq 1. \\ & x \geq 0 \end{cases} \qquad (D)\begin{cases} \min\limits_y & 1 \cdot y \\ \text{s.t.} & 0 \cdot y \geq 1. \\ & y \geq 0 \end{cases}$$

A student Alice saw this problem (P), she wrote down the symmetric dual problem (D) as shown above. She then solved both problems and found that $x^* = p^* = +\infty$, $y^* = d^* = 0$. She found that $p^* \not\leq d^*$ and said "I have proved that weak duality is wrong!". Comment on her conclusion. If she is correct, explain why. If she is incorrect, explain where is the error and give the correct result.

6 Optimal Transport (15 points)

Consider the following directed graph of a supply chain. There are two supply nodes S1, S2 and three demand nodes D1, D2, D3. The supply nodes can supply 75 and 125 amount of products, respectively. The demand nodes want 80, 70 and 50 amount of products, respectively. The table on the right shows the corresponding cost of transporting between a supply node and a demand note. For example, 682 refers to the cost of supplying one unit of product from S2 to D1.



	D1	D2	D3	
S1	464	654	513	75
S2	682	685	388	125
	80	70	50	

1. Solve the optimal transport problem for this graph.
2. Suppose the demand of D3 changed to 65, and now the network has more demand than supply. To solve this issue, the company buys supply by outsourcing with all the transport cost equal to 1000 per unit. Solve the optimal transport problem.

Administrative announcement

- If you haven't submit your course evaluation, go to `evaluate.uwaterloo.ca` and spend 5 minute to complete the course evaluation before August 5.
- This is to reminder you to submit the scribe note assignment (20% of the course grade point).

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