Solution to CO327 (2021Spring) Final Assignment

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1 Linear Programming (30 points)

Consider the following problem

$$\begin{array}{ll} \min & -2x_1 - x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 3 \\ & \left|x_1 + 4x_2\right| \leq 5 \\ & x_1 \geq 0 \\ & -x_2 \leq 0 \\ & |x_3| \geq 0 \end{array} \tag{1}$$

1. Convert (1) to canonical form

$$\begin{array}{ll} \max\limits_{\mathbf{x}} & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{array} \tag{1'}$$

2. Identify any redundant constraint in (1′).

3. Solve (1′) , give an optimal solution point $\mathbf{x}^*$.

4. State the active set S of $\mathbf{x}^*$. Based on $\mathbf{A}_S\mathbf{x}^* = \mathbf{b}_S$, what can you tell about the uniqueness of the solution to (1′)?

5. Convert (1′) to the following form

$$\begin{array}{ll} \max\limits_{\mathbf{x}} & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \tag{1''}$$

   Then derive the symmetric dual of (1″) and solve it. Verify the duality and the complementary slackness between the primal and the dual problems.

6. Now consider the following ILP problem denoted as $\mathcal{P}(\mathbf{c}, \mathbf{A}, \mathbf{b})$:

$$\mathcal{P}(\mathbf{c}, \mathbf{A}, \mathbf{b}) : \left\{ \begin{array}{ll} \min & -2x_1 - x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 3 \\ & x_1 + 4x_2 \leq 5 \\ & x_i \in \{0, 1\}, \text{for all } i \end{array} \right.$$

   - Show that $\mathbf{A}$ is not unimodular.
   - Solve $\mathcal{P}(\mathbf{c}, \mathbf{A}, \mathbf{b})$ and confirm that LP relaxation works for this IP.

- A student Alice said that "For $\mathcal{P}(\mathbf{c}, \mathbf{A}, \mathbf{b})$, the matrix $\mathbf{A}$ is not unimodular, however LP relaxation works for this problem, hence this contradicts to the lectures on the unimodularity, and therefore unidmoularity is wrong". Comment on her statement. Explain why even $\mathbf{A}$ is not unimodular, LP relaxation still works for $\mathcal{P}(\mathbf{c}, \mathbf{A}, \mathbf{b})$, and explain when will LP relaxation fail for $\mathcal{P}$.

7. Make it as simple as you can, make up a story of a LP problem that, when formulating such a problem will give raise to $(1'')$.

Sol:

1. Note that x_3 is redundant so it can be removed. 1 pt.

$$\begin{aligned} \max \quad & \begin{bmatrix} 2 \\ 1 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} 2 & 1 \\ 1 & 4 \\ -1 & -4 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 5 \\ 5 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \qquad (3 \text{ pt.})$$

2. $|x_3| \geq 0$ is redundant (if haven't remove it in part 1) 1 pt.
   $-1x_1 - 4x_2 \leq 5$ is redundant (because of $x_1 \geq 0, x_2 \geq 0$) 1 pt.

3. The solution is $x^* = [1.5, 0]^\top$. 1 pt.

4. (If x_3 is removed) The active set of $\mathbf{x}^*$ is $\{1, 5\}$ 2 pt.
   (If x_3 is not removed, add back the row index correspond to x_3.)
   The solution is non-unique 1 pt.

   (In fact, any point on $2x_1 + x_2 \leq 3$ is also a sol because $\mathbf{c}$ is exactly the normal to such hyperplane. The $\mathbf{A}_S\mathbf{x}^* = \mathbf{b}_S$ argument here is useless and to confuse student.)

5. The new problem (after redundant constraints removed)

$$\begin{aligned} \max \quad & \begin{bmatrix} 2 \\ 1 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \qquad (4 \text{ pt.})$$

   The symmetric dual

$$\begin{aligned} \min \quad & \begin{bmatrix} 3 \\ 5 \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned} \qquad (4 \text{ pt.})$$

   Sol: $\mathbf{y}* = [1, 0]^\top$. 1 pt.
   Strong duality is true: $\mathbf{c}^\top\mathbf{x}^* = 3 = \mathbf{b}^\top\mathbf{y}^*$. 1 pt.

   Complementary slackness : yes by checking $\mathbf{y}^* \geq \mathbf{0}$ vs $\mathbf{A}\mathbf{x}^* - \mathbf{b}$ and $\mathbf{x}^* \geq \mathbf{0}$ vs $\mathbf{A}^\top\mathbf{y}^* - \mathbf{c}$ 2 pt.
   (Note that it might seem not ture at first glance)

6. $\det(\mathbf{A}) = 7 \neq 1$ so $\mathbf{A}$ is not unimodular. 1 pt.

   LP relaxation works here just because $\mathbf{A}^{-1}\mathbf{b}$ is integer:

$$\mathbf{A}^{-1}\mathbf{b} = \frac{1}{7}\begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}\begin{bmatrix} \frac{12}{7} - \frac{5}{7} \\ \frac{-3}{7} - \frac{10}{7} \end{bmatrix} = \begin{bmatrix} \frac{7}{7} \\ \frac{7}{7} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{(2 pt.)}$$

   LP relaxation will not work in general if we have change $\mathbf{b}$ a little bit. Therefore this do not contradict with the theory of unimodularity. 3 pt.
   OR: integrality gap is zero for this problem / active constraint intersect at integer even though $\mathbf{A}$ is not unimodular

7. Any story that makes sense 3 pt.

2 ℓ_1 and ℓ_∞ norms (15 points)

Consider the following problem

$$\begin{array}{ll} \min & \|\mathbf{Ax}-\mathbf{b}\|_\infty + 2\|\mathbf{x}\|_1 \\ \text{s.t.} & \mathbf{0} \leq \mathbf{x} \leq \begin{bmatrix} 5 \\ 4 \end{bmatrix} \end{array}, \quad \mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ p \end{bmatrix} \qquad (2)$$

Solve (2) with $p = 2$, 20, 200, 2000. What did you observe? Explain what happened.

Sol. Introduce t and $\mathbf{u} \in \mathbb{R}^2$

$$\min_{\mathbf{x}} \ \|\mathbf{Ax}-\mathbf{b}\|_\infty + 2\|\mathbf{x}\|_1 = \min_{\mathbf{x},t,\mathbf{u}} \ t + 2\cdot\mathbf{1}^\top\mathbf{u}$$

Constraints

$$\mathbf{0} \leq \mathbf{x} \leq \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \ -t\mathbf{1} \leq \mathbf{Ax}-\mathbf{b} \leq t\mathbf{1}, \ -\mathbf{u} \leq \mathbf{x} \leq \mathbf{u}$$

Everything together

$$\min \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \\ t \\ u_1 \\ u_2 \end{bmatrix} \quad \text{s.t.} \begin{bmatrix} -\mathbf{I}_2 & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times2} \\ \mathbf{I}_2 & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times2} \\ -\mathbf{A} & -\mathbf{1}_{3\times1} & \mathbf{0}_{3\times2} \\ \mathbf{A} & -\mathbf{1}_{3\times1} & \mathbf{0}_{3\times2} \\ -\mathbf{I}_2 & \mathbf{0}_{2\times1} & -\mathbf{I}_2 \\ \mathbf{I}_2 & \mathbf{0}_{2\times1} & -\mathbf{I}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ t \\ u_1 \\ u_2 \end{bmatrix} \leq \begin{bmatrix} \mathbf{0}_{2\times1} \\ 5 \\ 4 \\ -\mathbf{b} \\ \mathbf{b} \\ \mathbf{0}_{2\times1} \\ \mathbf{0}_{2\times1} \end{bmatrix}, \text{optimal value} = p$$

Marking

- Reformulation: 6 pts
- Solution: 4 pts
- Explanation: 5 pts if using sensitivity and duality, 3 pts for other sols that make sense.

  Sensitivity: consider the perturbation on vector $\mathbf{b}$ with $\Delta = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}$. Based on the sensitivity anylsis on the dual form with dual cost $\mathbf{y}^\top(\mathbf{b}+\Delta)$, we can derive that the optimal (primal) cost has a sensitivity directly related to p.

3 Applications (20 points)

3.1 A story from One Thousand and One Nights/Arabia's night.

A king met 3 boys and told them about his mule. "What color is the mule?" asked one boy. "Either brown, black, or gray. Make a guess." answered the king. The boys made the following guesses:

Boy	1	2	3
Guess	Black	Either brown or gray	Brown

The king replied: "at least one of you guessed right, and at least one of you guessed wrong".

1. Formulate this problem as BIP.
2. Solve the BIP. Which boy guessed it right? What is the mule's color?

Solution The tricky part of this question is not to mix up mule's color and the boy's guess as a single variable.

Variables Let $x_{\text{black}}, x_{\text{brown}}, x_{\text{gray}} \in \{0,1\}$ be the variable of the mule's color, and let $y_1, y_2, y_3 \in \{0,1\}$ be correct/wrong for the guesses.

Objective function There is no objective function, or in other words, $\mathbf{c} = \mathbf{0}$

Explicit constraints "At least one boy guessed right" means

$$y_1 + y_2 + y_3 \geq 1$$

"At least one boy guessed wrong" means

$$\begin{aligned} & (1-y_1) + (1-y_2) + (1-y_3) \geq 1 \\ \iff & y_1 + y_2 + y_3 \leq 2 \end{aligned}$$

Implicit constraints "Exactly one color"

$$x_{\text{black}} + x_{\text{brown}} + x_{\text{gray}} = 1.$$

"If boy 1 is correct then mule is black"

$$x_{\text{black}} - y_1 = 0 \implies x_{\text{black}} = y_1 \text{ and we can replace } y_1 \text{ by } x_{\text{black}}$$

"If boy 3 is correct then mule is brown"

$$x_{\text{brown}} - y_3 = 0 \implies x_{\text{brown}} = y_3 \text{ and we can replace } y_3 \text{ by } x_{\text{brown}}$$

"If boy 2 is correct then mule is brown xor (exclusive or) gray"

$$y_2 \leq x_{\text{brown}} + x_{\text{gray}},\ y_2 \leq 2 - x_{\text{brown}} - x_{\text{gray}},\ y_2 \leq x_{\text{brown}} - x_{\text{gray}},\ y_2 \leq x_{\text{gray}} - x_{\text{brown}}$$

Combine everything

min	0	1 pt
s.t.	$x_{\text{black}} + x_{\text{brown}} + x_{\text{gray}} = 1$	1 pt
	$x_{\text{black}} + y_2 + x_{\text{brown}} \geq 1$	1 pt
	$x_{\text{black}} + y_2 + x_{\text{brown}} \leq 2$	1 pt
	$y_2 \leq x_{\text{brown}} + x_{\text{gray}}$	0.5 pt
	$y_2 \leq 2 - x_{\text{brown}} - x_{\text{gray}}$	0.5 pt
	$y_2 \leq x_{\text{brown}} - x_{\text{gray}}$	0.5 pt
	$y_2 \leq x_{\text{gray}} - x_{\text{brown}}$	0.5 pt
	$x_{\text{black}}, x_{\text{brown}}, x_{\text{gray}} \in \{0, 1\}$	2 pt

Sol: $\mathbf{x} = [1, 0, 0]$, the mule is black. 2 pt.

Note:

- Solving the problem using inverse arguments (work backward from trying each solution) get zero points because such approach is not solving the problem using BIP.
- Saying there is no solution without using proper BIP modeling also get zero points because the argument of no solution is not coming from BIP modeling but inverse arguments.
- If exclusive or is replaced by or in the modeling of y_2, also treated as correct (if the expressions are correct).

3.2 2-player zero-sum game

Two players: player 1 (P1) and player 2 (P2) play the Paper scissor rock game. (If you don't know this game, see `https://en.wikipedia.org/wiki/Rock_paper_scissors`.) From the perspective of P1, we have the following payoff table, where +1 means P1 is the winner, -1 means P1 loses and 0 means there is a draw.

	P2 Paper	P2 Scissor	P2 Rock
P1 Paper	0	- 1	+1
P1 Scissor	+1	0	-1
P1 Rock	-1	+1	0

Table 1: The payoff matrix of player 1 in the Paper Scissor rock game.

Now suppose P1 and P2 both use a probabilistic strategy to play this game: P1 will choose paper, scissor and rock with the probability x_P, x_S, x_R, respectively, and P2 will choose paper, scissor and rock with the probability y_P, y_S, y_R, respectively. Consider the situation that the two players will play the same game many many times. Use the theory of 2-player zero-sum game, prove that in this situation, the best strategy for P1 is to choose paper, scissor and rock with equal probability. In other words, prove that randomly selecting paper, scissor and rock with equal probability is the best way to play this game in the long run. (Hint: if you use MATLAB to solve the LP problem, use `linprog(c,A,b,Aeq,beq,l)`)

Problem formulation:

$$\begin{aligned} \max \quad & x \\ \text{s.t.} \quad & x_i \geq 0,\ i \in \{1,2,3\} \\ & x_1 + x_2 + x_3 = 1 \\ & -x_2 + x_3 \geq x \\ & x_1 - x_3 \geq x \\ & -x_1 + x_2 \geq x \end{aligned}$$

In LP form

$$\max \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x^+ \\ x^- \end{bmatrix} \quad \text{s.t.} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x^+ \\ x^- \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & 1 & -1 \\ -1 & 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x^+ \\ x^- \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x^+ \\ x^- \end{bmatrix} = 1$$

`linprog(-c,A,b,Aeq,beq,zeros(5,1))` gives [0.3333, 0.3333, 0.3333, 0, 0].

10 pt: correctly LP model 9, solution 1

4 Production economics and labour allocation (15 points)

In this problem you will solve a simplified model of production economics proposed by a Nobel Prize laureate. You are the president of Canada. Canada's economy encompasses a network of interdependent industries.

We now explain the terms. **Good**: Each industry both produces and consumes "goods". E.g., the steel industry consumes coal to manufacture steel. **Industry**: Each industry requires different resources per unit production. E.g., one industry for producing steel starts with iron ore while another makes use of scrap metal. **Labour**: Canada is rich in natural resources so you can assume labour is the only limiting factor. Your task is to develop a model to decide how the labour force should be allocated among industries.

Production matrix: Each industry produces a single good and may consume others. There are M goods, indexed $i = 1, \ldots, M$. Each can be produced by one or more industries. There are a total of $N \geq M$ industries, indexed $j = 1, \ldots, N$. Each jth industry produces $A_{ij} > 0$ units of some ith good per unit of labour. For each $k \neq i$, this jth industry may consume some amount of good k per unit labour, denoted by $A_{kj} \leq 0$. Note that A_{kj} is nonpositive; if it is a negative number, it represents the quantity of good k consumed per unit labour allocated to industry j. The productivity and resource requirements of all industries are therefore captured by a matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$ in which each column has exactly one positive entry and each row has at least one positive entry. This matrix $\mathbf{A}$ is called the production matrix.

Modelling the constraints Suppose we have a total of one unit of labour to allocate over the next year. Let $\mathbf{x} \in \mathbb{R}^N$ be our allocation among the N industries, where entries of $\mathbf{x}$ refer to the fraction of labour allocated to each industry.

Q1. Write down the constraint associated with x.

Q2. Using A and x, Write down the quantity of each of the M goods produced.

The objective function The objective is be to optimize the social welfare. Suppose the amount society values each unit of each ith good is $c_i > 0$, regardless of the quantity produced. Suppose our objective function is the social welfare generated linearly by production of goods of all industries.

Q3. Write down the objective function in terms of $\mathbf{c}, \mathbf{A}$ and $\mathbf{x}$.

Q4. Write down the whole optimization problem.

Q5. An example Solve the labour force allocation problem for the following table. There are 3 type of goods and 3 type of industries. How should you distribute the labour force? What is the final amount of production for each type of goods? What is the final total social welfare?

Industry		1	2	3	Society value
	1	0.4	-0.1	0	1
Goods	2	-0.1	0.8	-0.1	3
	3	-0.1	-0.5	0.5	2

1. $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{1}^\top \mathbf{x} = 1$ 2 pt.
2. $\mathbf{A}\mathbf{x}$ 1 pt.
3. $\max \mathbf{c}^\top \mathbf{A}\mathbf{x}$ 3 pt.

$$\begin{array}{rll} \max & \mathbf{c}^\top \mathbf{A}\mathbf{x} & \text{maximize total social welfare} \quad 1 \text{ pt.} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{0} & \text{nonnegative amount of goods produced} \quad 3 \text{ pt.} \\ & \mathbf{x}^\top \mathbf{1} = 1 & \text{total amount of labour is one unit} \quad 1 \text{ pt.} \\ & \mathbf{x} \geq \mathbf{0} & \text{nonnegative amount of labour allocated to all technologies} \quad 1 \text{ pt.} \end{array}$$

$\mathbf{x}^* = [0.1087, 0.4348, 0.4565]$. $\mathbf{A}\mathbf{x} = [0, 0.2913, 0]$, final total social welfare $= \mathbf{c}^\top \mathbf{A}\mathbf{x} = 0.8739$ 3 pt.

5 Duality of LP (25 points)

Notation In this question,

- $\mathbf{x}^*$ denotes the optimal solution of the primal problem
- $p^* = \mathbf{c}^\top \mathbf{x}^*$ denotes the optimal primal cost value / optimal objective function value of the primal problem
- $\mathbf{y}^*$ denotes the optimal solution of the dual problem
- $d^* = \mathbf{b}^\top \mathbf{y}^*$ denotes the optimal dual cost value / optimal objective function value of the dual problem

The problem Consider the following LP of single scalar variable: maximize cx subject to $ax \leq b$ and $x \geq 0$ with $a = 0, c = b = 1$. That is,

$$(P) \begin{cases} \max\limits_{x} & 1 \cdot x \\ \text{s.t.} & 0 \cdot x \leq 1. \\ & x \geq 0 \end{cases} \qquad (D) \begin{cases} \min\limits_{y} & 1 \cdot y \\ \text{s.t.} & 0 \cdot y \geq 1. \\ & y \geq 0 \end{cases}$$

A student Alice saw this problem (P), she wrote down the symmetric dual problem (D) as shown above. She then solved both problems and found that $x^* = p^* = +\infty$, $y^* = d^* = 0$. She found that $p^* \not\leq d^*$ and said "I have proved that weak duality is wrong!". Comment on her conclusion. If she is correct, explain why. If she is incorrect, explain where is the error and give the correct result.

Alice is wrong. 2 pt.

The constraint $0 \cdot x \leq 1$ have to be removed from (P) 2 pt.

therefore

$$(P) \begin{cases} \max_x & 1 \cdot x \\ \text{s.t.} & x \geq 0 \end{cases} \iff \begin{cases} \max_x & 1 \cdot x \\ \text{s.t.} & -x \leq 0 \end{cases} \iff \begin{cases} \max & \begin{bmatrix} 1 \\ -1 \end{bmatrix}^\top \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \\ \text{s.t.} & \\ & [-1\ 1] \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \leq 0 \\ & \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \geq 0 \end{cases} \qquad (10 \text{ pt.})$$

This (P) is unbounded therefore thee corresponding symmetric dual (D) is infeasible, which is true: 2+2 pt.

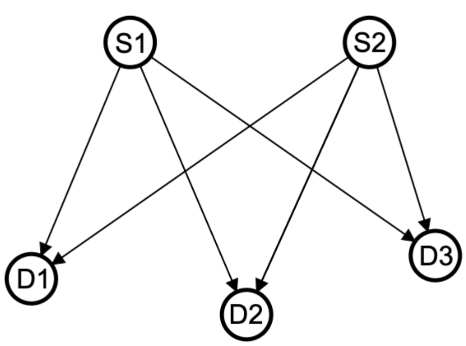
$$(D) \begin{cases} \min & 0 \cdot y \\ \text{s.t.} & \begin{bmatrix} -1 \\ 1 \end{bmatrix} y \geq \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ & y \geq 0 \end{cases} \qquad (5 \text{ pt.})$$

there is no y such that (D) is feasible. 2 pt.

Note: full marks only give to those who correctly reformulate (P) and (D). If only saying "(D) is infeasible" without showing the correct reformulation will only get 20 pts most.

6 Optimal Transport (15 points)

Consider the following directed graph of a supply chain. There are two supply nodes S1, S2 and three demand nodes D1, D2, D3. The supply nodes can supply 75 and 125 amount of products, respectively. The demand nodes want 80, 70 and 50 amount of products, respectively. The table on the right shows the corresponding cost of transporting between a supply node and a demand note. For example, 682 refers to the cost of supplying one unit of product from S2 to D1.



	D1	D2	D3	
S1	464	654	513	75
S2	682	685	388	125
	80	70	50	

1. Solve the optimal transport problem for this graph.
2. Suppose the demand of D3 changed to 65, and now the network has more demand than supply. To solve this issue, the company buys supply by outsourcing with all the transport cost equal to 1000 per unit. Solve the optimal transport problem.

$$\begin{array}{ll} \min & \sum_{ij} c_{ij}x_{ij} \\ \text{s.t.} & \sum_j x_{ij} = s_i \ i \in \{1,2\} \\ & \sum_i x_{ij} = d_j \ j \in \{1,2,3\} \\ & \mathbf{x}_{ij} \geq 0 \end{array}, \mathbf{C} = \begin{bmatrix} 464 & 654 & 513 \\ 682 & 685 & 388 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}, \mathbf{s} = \begin{bmatrix} 75 \\ 125 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} 80 \\ 70 \\ 50 \end{bmatrix}$$

$\mathbf{x}^* = [75, 5, 0, 70, 0, 50]$, cost $= 105560$
9 pt. in total

Part 2: add a new supply node S3 ("white hole") 1 pt.
with supply value $= 15$ (the insufficient amount of the original supply) 2 pt.
with cost 1000 to all demand nodes 2 pt.

$$\begin{array}{ll} \min & \sum_{ij} c_{ij}x_{ij} \\ \text{s.t.} & \sum_j x_{ij} = s_i \ i \in \{1,2,3\} \\ & \sum_i x_{ij} = d_j \ j \in \{1,2,3\} \\ & \mathbf{x}_{ij} \geq 0 \end{array}, \mathbf{C} = \begin{bmatrix} 464 & 654 & 513 \\ 682 & 685 & 388 \\ 1000 & 1000 & 1000 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \mathbf{s} = \begin{bmatrix} 75 \\ 125 \\ 15 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} 80 \\ 70 \\ 65 \end{bmatrix}$$

$\mathbf{x}^* = [75, 5, 0, 0, 55, 15, 0, 65, 0]$, cost $= 116105$ 1 pt.

Administrative announcement

- This is to reminder you to submit the scribe note assignment (20% of the course grade point).

-END-