CO327 (2021Spring) Midterm Assignment

Lecturer: Andersen Ang

July 1, 2021

Instructions

- In your PDF, write down your name and your student ID.
- You can use whatever method to generate your PDF (LaTeX is not necessary, but it is preferred). **Make your PDF size small**.
- Use of MATLAB is allowed. If you use MATLAB to solve a problem, **you need to provide with the MATLAB code**.
- Zip all your files into one single zip file. Name your zip as "Your student ID" + "midterm".zip and submit to the dropbox in Waterloo LEARN.
- Submission deadline
  - July-5 (NEW!) 11:55 (EDT).
  - Late submission get zero point.
  - Bonus: Those who submit it before July-3 11:55 (EDT) will get +4%.
    Those who submit it before July-4 11:55 (EDT) will get +2%.
  - If you re-submit: everything will be based on your newest submission, and your previous submission file will be ignored.
- Try your best in answering all the questions. Most of the questions in this midterm are similar to those discussed during class. However, there are some questions that are "hard", due to two reasons:
  1. some questions are hard because they are tricky.
  2. some questions are seemingly hard because they are designed to test your mathematical skills in facing unfamiliar cases.

  Do not left blank on questions that you think you don't know how to solve them. Try your best to solve them: at least say something / describe what you can do to solve them. These might help you get some points and are much better than just left it blank.

1 Cargo plane loading (15 points)

You own a logistic company and you have a cargo plane with 1000 meter^3 of space available. Five suppliers approach you and want you to deliever their cargo to the destination. After negotiating with these suppliers, you can choose from a sets of 5 cargos to load the plane. For each cargo, the volume and profit expected from shipping this cargo are listed.

Cargo type	Volume	Profit
1	410	200
2	600	60
3	220	20
4	450	40
5	330	30

Scenario 1 Now, suppose there are at most 1 cargo available from the suppliers. Suppose we are allowed to take any portion of each cargo up to the total volume, with the profit adjusted accordingly. That is, if you take half of the cargo, it yields half the profit. Model such cargo plane loading problem as an optimization problem. Solve this problem by MATLAB.

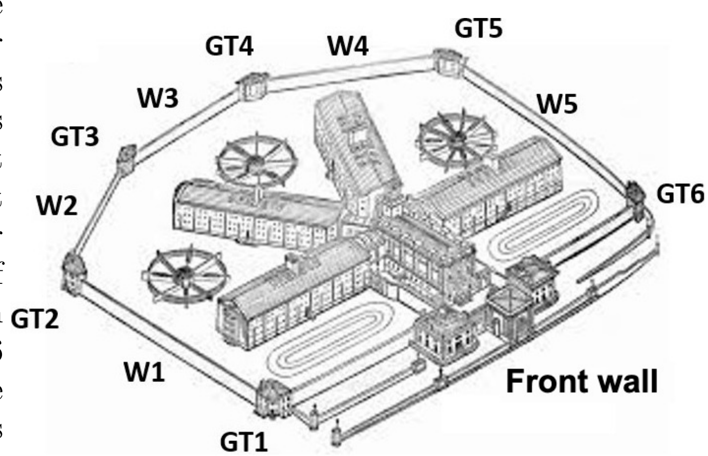
Scenario 2 Now, suppose there are at most 1 cargo available from the suppliers. Suppose we must take all of the cargo or none of it. Model such cargo plane loading problem as an optimization problem. Solve this problem by brute force (enumerating all the 2^5 possible solutions and find the best one).

Comparing the scenarios Compare the optimal profit obtained in these scenarios, explain why they are different and give an explanation of their relationship.

2 Integer programming (29 points)

2.1 Well-guarded prison (10 points)

You are the warden of Pentonville prison, see the figure on the right, where GT stand for guard tower and W stand for wall. There are six Guard Towers (GT), labeled as GT1 to GT6, and there are 6 walls (W), labeled as W1 to W5, together with the front wall. In order to prevent "prison break", you put soldiers on towers to watch the walls. Each soldier in each tower can watch the two adjacent walls of that tower. For example, each soldier on GT3 can watch wall 2 and wall 3, and each soldier on GT6 can watch W5 and the front wall. Your task as the warden is to determine an arrangement of soldiers such that each wall is well guarded by soldiers.



Front wall with fortification You build the front wall of the prison with fortification, therefore it requires fewer soldiers to watch. Determine an arrangement of the minimum number of soldier such there are at least 5 soldiers watching each wall (W1-W5), and there are at least 2 soldiers watching the front wall. What you need to do: write down the optimization problem, solve it by MATLAB, show the solution.

Front wall with very strong fortification You build the front wall of the prison with very strong fortification, therefore it requires no soldier to watch. Determine an arrangement of the minimum number of soldiers such there are at least 5 soldiers watching each wall (W1-W5). What you need to do: write down the optimization problem, solve it by MATLAB, show the solution.

Relaxation Describe what happened to the (optimal) solution when we relax the constraint on number of soldiers required to watch the front wall from 2 to 0.

2.2 Rental property decision problem (7 points)

You own a rental property and you rent out the property for 10 weeks (consecutively) in total. You have received 6 requests specifying which week each renter would like to start and how many weeks they want to stay (duration). The rate of renting the property is 100 dollar per week normally, but if anyone rents the property for at least 4 weeks, then they are charged 70 per week. Decide which requests to accept in order to maximize your income. Formulate this problem as an integer program. State clearly the decision variable(s), the cost function and the constraint(s). Solve the optimization problem.

person	start week	duration
A	1	1
B	1	4
C	3	2
D	5	3
E	7	2
F	9	1

2.3 Solving IP (12 points)

Consider the following BIP:

$$\begin{aligned} \min \quad & 5x_1 + 4x_2 + 6x_3 + 2x_4 + 4x_5 \\ \text{s.t.} \quad & x_1 + x_2 \geq 1 \\ & x_2 + x_5 \geq 1 \\ & x_2 + x_3 + x_4 \geq 1 \\ & x_2 + x_3 + x_4 + x_5 \geq 1 \\ & x_i \in \{0, 1\} \text{ for all } i \end{aligned}$$

1. Is (are) there any redundant constraint(s)? If yes, simplify the problem.
2. Is this IP solvable? If yes, solve this IP. If no, explain why.
3. Make it as simple as you can, make-up a story of a location problem that, when formulating such a problem will give raise to this IP. Draw the map of the location problem.

3 Reformulation (30 points)

3.1 Reformulate the LP to (7 points)

Reformulate the problem

$$\begin{aligned} \min \quad & 3x_1 - x_2 \\ \text{s.t.} \quad & -x_1 + 6x_2 - x_3 + x_4 \geq -3 \\ & 7x_2 + x_4 = 5 \\ & x_3 + x_4 \leq 2 \\ & -1 \leq x_2 \\ & x_3 \leq 5 \\ & |x_4| \leq 2. \end{aligned}$$

to a LP in the form

$$1.\ \max_{\mathbf{y}}\ \mathbf{c}^\top\mathbf{y}\ \text{ s.t. }\ \mathbf{A}\mathbf{y}\le\mathbf{b},\mathbf{y}\ge\mathbf{0}\qquad 2.\ \max_{\mathbf{y}}\ \mathbf{c}^\top\mathbf{y}\ \text{ s.t. }\ \mathbf{A}\mathbf{y}=\mathbf{b},\mathbf{y}\ge\mathbf{0}$$

Then solved these problem by using `linprog` in MATLAB. If the problem has a solution, show the solution. If the problem has no solution, explain why.

3.2 Reformulate the problems as LPs (9 points)

$$1.\ \min\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_\infty,\qquad 2.\ \min\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_1\qquad 3.\ \min\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_1\ \text{s.t.}\ \|\mathbf{x}\|_\infty\le 1$$

3.3 Solve this problem (4 points)

Find the scalar x that is the minimizer of the problem

$$\min_x\ \left(\ \Big|4-\max\{1,x\}\Big|+\Big|3-\max\{2,x\}\Big|\ \right).$$

If the problem has exactly one solution, show the solution. If the problem has more than one solution, show all the solutions. If the problem has no solution, explain why.

3.4 Reformulate the IP as BIP / 0-1 IP (9 points)

Given the hint that $3=1+2$, reformulate the following integer program problem into a binary integer program.

$$\begin{array}{ll}\min & x_1+x_2+x_3\\ \text{s.t.} & x_1+x_2\le 10\\ & x_2+x_3\le 10\\ & x_1+x_3\le 10\\ & 0\le x_i\le 3,\quad i\in\{1,2,3\}\\ & x_i \text{ is integer}\end{array}$$

Note: you do not need to solve the integer program.

3.5 Reformulate to QP (5 points)

Reformulate the following problem to QP in the form of $\min\mathbf{x}^\top\mathbf{Q}\mathbf{x}+\mathbf{p}^\top\mathbf{x}\ \text{ s.t. }\ \mathbf{A}\mathbf{x}\le\mathbf{b}$.

$$\min\frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2+\|\mathbf{C}\mathbf{x}-\mathbf{d}\|_1$$

Note: you do not need to solve the quadratic program.

4 Cpacitated Location Planning (30 points)

You are going to formulate and solve two facilities location planning problems. These problems are variants of the facilities location planning problems discussed in class. In these problems, the facilities have a maximum capacity of demand that the facility is able to serve.

The problem is presented by an undirected graph $G = (V, E)$, with V denotes the set of vertices and E denotes the set of edge. Here V has two subset : $V = \{I, J\}$, where the demand nodes are represented by a set of vertices $i \in I$, the possible locations of the facilities are given by the set of vertices $j \in J$, and edges $e_{i,j} \in E$ only exist between vertices from $i \in I$ to those in $j \in J$. Furthermore, we assign positive weights to the edges $d_{i,j} \geq 0$, which represents a distance between vertices i and j (Note that it is possible to have zero distance between a demand node and a possible facility location.). We assign positive weights to the demand nodes h_i for $i \in I$, and this represents the amount of demand at a particular node. Moreover, each facility has a maximum capacity C of demand that can be served.

4.1 A graph

Given $\mathbf{I} = \{1, 2, 3, 4, 5\}$, $\mathbf{J} = \{1, 2, 3\}$, $\mathbf{h} = \{1, 1, 1, 1, 1\}$, $C = 2$ and

$$\mathbf{E}\Big\{(1,1),(2,1),(3,1),(4,1),(5,1),(1,2),(2,2),(3,2),(3,3),(4,3),(5,3)\Big\}, \quad \mathbf{d} = \begin{bmatrix} 1 & 1 & * \\ \frac{1+\sqrt{5}}{2} & 0 & * \\ 0 & \frac{1+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & * & 1 \\ \frac{1+\sqrt{5}}{2} & * & 0 \end{bmatrix}$$

where $d_{ij} = *$ means people cannot go from i to j directly (you can take it as $+\infty$).

Draw the graph and label the demand node i and possible facility location j. What is the shape of this graph?

4.2 Capacitated p-Median Problem

In this problem, you wish to place p facilities to minimize the (demand-weighted) average distance between a demand node and the location in which a facility was placed, subject to the constraints on the capacity of the facilities. Formulate this capacitated p-median problem as the an ILP (Integer Linear Program). Solve this problem for $p = 1$.

4.3 Capacitated p-Center problem

Now, you want to place p facilities to minimize the **maximum distance** between any demand node and its servicing facility, under the same constraints (capacity of the facilities.) Formulate this capacitated p-center problem as the an ILP. Solve this problem for $p = 1$.

5 Project management: Activity Scheduling (16 points)

You are going to formulate a complicated problem that is used in real-life project management.

Introduction You are the manager of a big construction project:

- The **project** involves n **activities** (tasks, operations) to be performed **in parallel**, subject to a set of **temporal constraints**.
- The **start-finish** constraints specify the minimum time lag between the start of one activity and the finish of another. Each activity finishes as soon as its start-finish constraints are fulfilled.
- The **release time** and **release deadlines** define the earliest and latest allowed times (time window) for activities to start.
- The **flow-time** of an activity is the length of the time interval between its start and finish
- The **maximum deviation** of the start time is the difference between the latest and the earliest start times over all activities.
- The **scheduling objectives** are to minimize both the maximum flow-time and maximum deviation of the start time over all activities.

Overall hint When answering the questions below, you may find that your formulation looks weird. In fact, the correct formulation of this problem does look weird, therefore do not afraid of weird-looking equation.

5.1 Modelling the start time and its deviation

Now, for each activity $i = 1, 2, \ldots, n$, let x_i be the unknown start time of the ith activity.

1. Write down the expression of the latest start time over all activities.
2. Write down the expression of the earliest start time over all activities.
3. Hence, write down the expression of the maximum deviation of the start time of activities.

5.2 Modelling the flow-time

Now, together with the previously defined notation, let y_i be thee unknown finish time for each activity $i = 1, 2, \ldots, n$.

1. Write down the expression of the flow-time of activity i
2. Write down the expression of the maximum flow-time over all activity

5.3 Modelling the release time and release deadline constraints

Now, together with the previously defined notation, for each activity $i = 1, 2, \ldots, n$, let g_i be the given release time and h_i be the given relese deadline. Write down the release time and release deadline constraints

5.4 Modelling the start-finish constraints

Now, together with the previously defined notation, let a_{ij} be he given minimum possible time lag between the start of activity j and the finish of i. Write down the start-finish constraints.

5.5 Modelling the cost function

The scheduling objectives are to minimize both the maximum flow-time and maximum deviaation of the strt time over all activities. Write down the objective function.

5.6 Modelling the project scheduling problem

Write down the whole optimization problem.

6 Minesweeper (25 points)

Introduction (from wikipedia) Minesweeper is a single-player puzzle video game. The objective here is to clear a rectangular board containing hidden "mines" or bombs without detonating any of them, with the helps from the clues about the number of neighboring mines in each field. An example is shown in Fig.1.

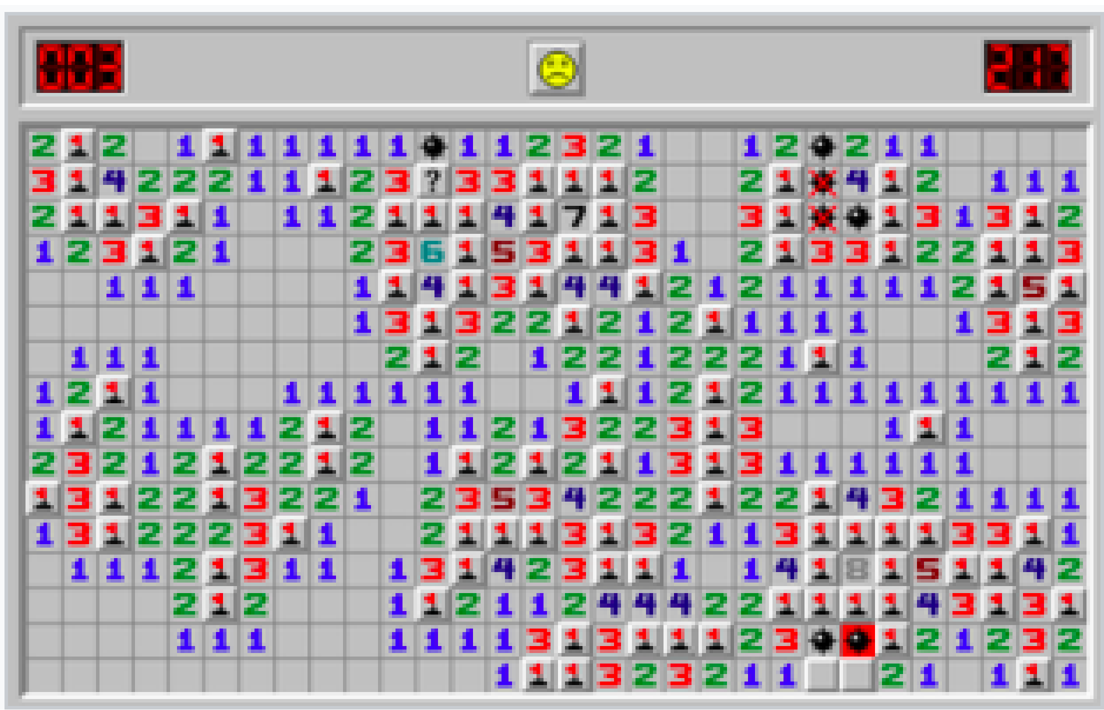
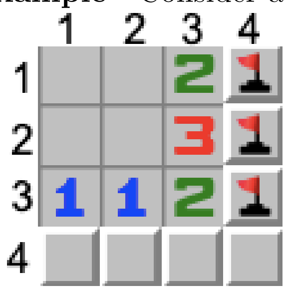


Figure 1: A typical minesweeper game with its commonly used style. This level (16 x 30 grid with 99 mines) is the "Expert" setting in typical implementations. Many boards have unavoidable guesses, causing lost games. Image from: wikipedia.

(If you have never played minesweeper, then before reading the following looooong chunk of words, the best to understand the mechanism of minesweeper is to try it yourself.)

The game description (Raphaël Collet): you are given a board represents a mine field, with mines hidden under the squares. Each square on the board may hide at most one mine. The total number of mines is known by the player. The game consists in finding the mines without making them explode. You get new hints each time you uncover a non-mined square. A move consists in uncovering a square. If the square holds a mine, the mine explodes and the game is over. Otherwise, a number in the square indicates how many mines are held in the surrounding squares, which are the adjacent squares in the eight directions north, north-east, east, south-east, south, southwest, west, and north-west. The goal of the game is to uncover all the squares that do not hold a mine.

A concrete example Consider a 4-by-4 mine field. In this example, there are 4 mines in the field.



$$F = \begin{bmatrix} 0 & 0 & 2 & ? \\ 0 & 0 & 3 & ? \\ 1 & 1 & 2 & ? \\ ? & ? & ? & ? \end{bmatrix} \qquad X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

Refer to the figure, a 4×4 board is given and some squares are uncovered. Each square is identified by its coordinates $(row, column)$. We have the following:

1. The squares (1,1), (1,2), (2,1), (2,2) have already been played, and have no mine in their respective surrounding squares.
2. The squares (3,1), (3,2) have been played, too, and are surrounded by one mined square each.
3. The squares (1,3) and (3,3) each have two mines in their neighborhood.
4. The square (2,3) has three mines around it.

In this example, the player might deduce from (2,3) that (1,4), (2,4) and (3,4) are mined (the flag), and by (3,3) that (4,1), (4,3) and (4,4) are safe move and lastly by (3,1) and (3,2) that (4,1) is mined.

Such board can be represented mathematically as a matrix F. And the solution to the minesweeper problem can be represented by a binary matrix $X = x_{ij}$: value 1 means that the corresponding (i, j) square is mined, while 0 means a safe square. Solving this minesweeper problem corresponds to find the x_{ij} value subject to the constraints of the game.

6.1 The example

- In the example, the variable X is being solved by deduction using the information (the list 1,2,3,4) and the game rule (there are totally 4 mines in the field, the number of the square gives the hint of how many mines are held in the surrounding squares). By using these information, mathematically we uncover the unknown X as

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 0 & 0 & 0 & x_{14} \\ 0 & 0 & 0 & x_{24} \\ 0 & 0 & 0 & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \xrightarrow{2,3,4} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

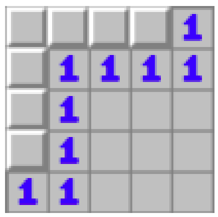
Now, write down mathematically the constraints 2,3,4 in the list.

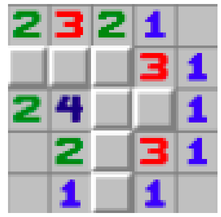
6.2 Constraint Satisfaction Problem

Such a problem is an example of *Constraint Satisfaction Problem* (CSP): there is there is no cost function. Mathematically,

$$\text{minimize } 0 \text{ subjec to constraints on } x$$

Now, write down the full optimization problem for the following two minesweeper problems. Are these minesweeper problems solvable? If yes, solve these minesweeper problems and show the solution. If no, explain why they are not solvable.





6.3 More questions

Solve the following minesweeper problem. You do not need to show the full optimization problem, you just need to show where is (are) the mine(s).

$$F = \begin{bmatrix} ? & 1 & ? & ? & ? \\ ? & ? & 3 & 3 & ? \\ 3 & ? & 4 & 2 & ? \\ ? & ? & ? & ? & 0 \\ ? & 2 & ? & ? & ? \end{bmatrix}$$

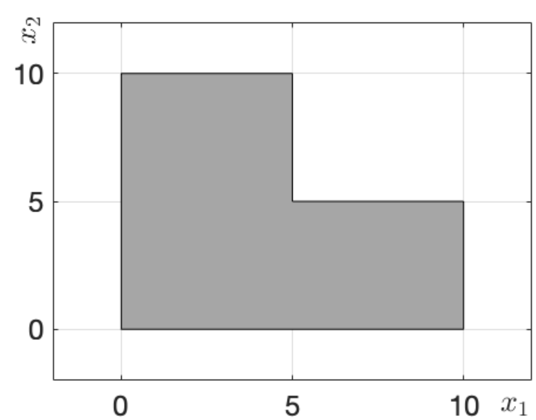
7 Simple LP (10 points)

Consider a simple LP in $\mathbb{R}^2$ (2-dimensional space).

1. Draw the feasible region of $0 \leq x_1 \leq 10$, $0 \leq x_2 \leq 10$.
2. Write down the inequalities correspond to the feasible region of the figure in the right (the gray region). Hint: introduce extra variables.
3. Let **C** represents the feasible region of the figure in the right. Design a simple LP

   $$\max\ \mathbf{c}^\top \mathbf{x} \,\text{s.t.}\, \mathbf{x} \in \mathcal{C}$$

   such that when solving this LP, there are exactly TWO solutions for $\mathbf{x} \in \mathbb{R}^2$.



8 Marriage Matchmaking (15 points)

There are n men $i \in \{1, 2, \ldots, n\}$ and n women $j \in \{1, 2, \ldots, n\}$ do not want to be single anymore and all of them want to have a marriage. You are a professional matchmaker and your job is to match these people together.

- For simplicity, assume all these people are heterosexual (no gay nor lesbian), and each person has a preference list of the opposite sex.
- Suppose M_{ij} is the score of woman j given by man i, and W_{ij} is the score of man i given by woman j. Here, a higher score means a higher preference.
- A marriage, is a one-to-one correspondence between a man and a woman (that is, we do not consider the case of man-man-woman nor woman-woman-man marriage).
- An unstable marriage is that if there exist two marriage couples (i_1, j_1) and (i_2, j_2) such that man i_1 desires j_2 more than j_1 and woman j_1 desires i_2 more than i_1. In this case (i_1, j_1) is an unhappy couple.

The questions

1. Given the matrices M and W, describe how you will solve such marriage matchmaking problem (matching couples such that there is no unstable marriage) by formulating an optimization problem. Describe what are the decision variables, what is the objective function, what are the constraints, and what kind of optimization problem it is.
2. Now, consider the cases that some people are gay / lesbian / bisexual. Describe how you will solve the marriage matchmaking problem (matching couples such that there is no unstable marriage), describe how you will reformulate the optimization problem in part 1 in this new case.
3. Now, suppose you are in a place where bigamy (multiple marriage in the form of man-man-woman or man-woman-woman) is allowed. Describe how you will solve the marriage matchmaking problem (matching couples such that there is no unstable marriage), describe how you will reformulate the optimization problem in part 1 in this new case.

9 About the course (10 points)

Please answer the following question honestly. This will help improve this course.

- What do you think about the difficulty of the course so far? For example, you think the difficulty of the assignments is (Choose one):

  too easy, easy, acceptable, hard, too hard

- What do you think about the work load of assignments so far? Choose one:

  want more assignment, light, acceptable, little heavy, too many assignments

- You might feel that the questions in the assignments / lecture material are not like those textbook questions you typically see in other courses. Compared with other courses, what do you think about these type of questions / material used in this course? Choose one

  very interesting, more fun than usual, no comment, boring as usual, very boring

- Name one thing you like about the course.
- Name one thing you don't like about the course.

Administrative announcement

- A reminder: start working on Assignment 5 early.
- Scribing: The theory part of the course is coming, these lectures will have no notes and you are required to take notes yourself and submit it. Scribe notes contribute part of your final grade.
- All PDFs after this midterm (assignments, scribe notes) need to be generated using LaTeX. Play with LaTeX early to familiarize yourself with it.

-END-