Solution to CO327 (2021Spring) Midterm Assignment

Lecturer: Andersen Ang

July 11, 2021

1 Cargo plane loading (15 points)

You own a logistic company and you have a cargo plane with 1000 meter^3 of space available. Five suppliers approach you and want you to deliever their cargo to the destination. After negotiating with these suppliers, you can choose from a sets of 5 cargos to load the plane. For each cargo, the volume and profit expected from shipping this cargo are listed.

Cargo type	Volume	Profit
1	410	200
2	600	60
3	220	20
4	450	40
5	330	30

Scenario 1 Now, suppose there are at most 1 cargo available from the suppliers. Suppose we are allowed to take any portion of each cargo up to the total volume, with the profit adjusted accordingly. That is, if you take half of the cargo, it yields half the profit. Model such cargo plane loading problem as an optimization problem. Solve this problem by MATLAB.

Scenario 2 Now, suppose there are at most 1 cargo available from the suppliers. Suppose we must take all of the cargo or none of it. Model such cargo plane loading problem as an optimization problem. Solve this problem by brute force (enumerating all the 2^5 possible solutions and find the best one).

Comparing the scenarios Compare the optimal profit obtained in these scenarios, explain why they are different and give an explanation of their relationship.

Sol Let

- V_j be the volume of the jth cargo
- P_j the profit
- $V = 1000$ the total volume available in the cargo plane.

Introduce the decision variable x_j to represent the amount of the jth cargo that we decide to take.
The optimization problem is

$$\max \quad \mathbf{p}^\top \mathbf{x} \text{ s.t. } \mathbf{V}^\top \mathbf{x} \leq V, \quad \mathbf{x} \geq \mathbf{0} \quad \text{and } \mathbf{x} \in \mathcal{C}$$

where

$$\mathbf{p} = \begin{bmatrix} 200 \\ 60 \\ 20 \\ 40 \\ 30 \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} 410 \\ 600 \\ 220 \\ 450 \\ 330 \end{bmatrix} \qquad V = 1000$$

- scenario 1: $\mathbf{0} \leq \mathbf{x} \leq \mathbf{1}$
  Optimal solution: $\mathbf{x} = [1, 0.9833, 0, 0]^\top$, optimal cost $= 259$
- scenario 2: $\mathbf{x} \in \{0,1\}^5$
  Optimal solution: $\mathbf{x} = [1, 0, 1, 0, 1]^\top$, optimal cost $= 250$
- Scenario 1 has **less restriction** than scenario 2 so it has a higher optimal value (more flexible / more freedom to optimize).

Marking:

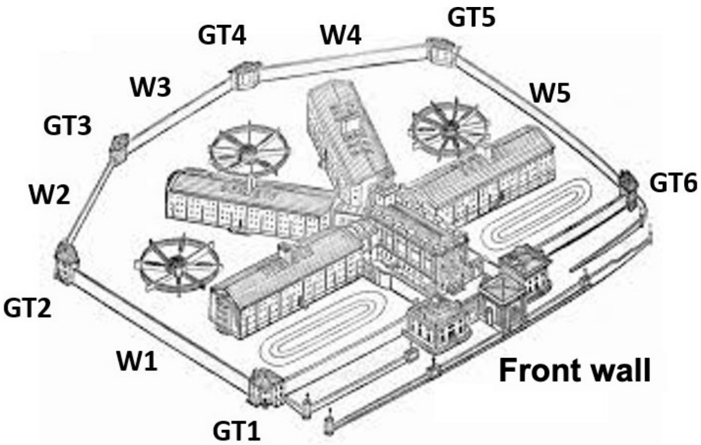
- 3 (optimization problem)
- 3 (scenario 1 sol)
- 3 (scenario 2 sol)
- 3 (scenario 2 brute force)
- 3 (comparison with sound explanation)

cargo choice 1	2	3	4	5	profit
0	0	0	0	0	0
0	0	0	0	1	30
0	0	0	1	0	40
0	0	0	1	1	70
0	0	1	0	0	20
0	0	1	0	1	50
0	0	1	1	0	60
0	0	1	1	1	90
0	1	0	0	0	60
0	1	0	0	1	90
0	1	0	1	0	0
0	1	0	1	1	0
0	1	1	0	0	80
0	1	1	0	1	0
0	1	1	1	0	0
0	1	1	1	1	0
1	0	0	0	0	200
1	0	0	0	1	230
1	0	0	1	0	240
1	0	0	1	1	0
1	0	1	0	0	220
1	0	1	0	1	250
1	0	1	1	0	0
1	0	1	1	1	0
1	1	0	0	0	0
1	1	0	0	1	0
1	1	0	1	0	0
1	1	0	1	1	0
1	1	1	0	0	0
1	1	1	0	1	0
1	1	1	1	0	0
1	1	1	1	1	0

2 Integer programming (29 points)

2.1 Well-guarded prison (10 points)

You are the warden of Pentonville prison, see the figure on the right, where GT stand for guard tower and W stand for wall. There are six Guard Towers (GT), labeled as GT1 to GT6, and there are 6 walls (W), labeled as W1 to W5, together with the front wall. In order to prevent "prison break", you put soldiers on towers to watch the walls. Each soldier in each tower can watch the two adjacent walls of that tower. For example, each soldier on GT3 can watch wall 2 and wall 3, and each soldier on GT6 can watch W5 and the front wall. Your task as the warden is to determine an arrangement of soldiers such that each wall is well guarded by soldiers.



Front wall with fortification You build the front wall of the prison with fortification, therefore it requires fewer soldiers to watch. Determine an arrangement of the minimum number of soldier such there are at least 5 soldiers watching each wall (W1-W5), and there are at least 2 soldiers watching the front wall. What you need to do: write down the optimization problem, solve it by MATLAB, show the solution.

Front wall with very strong fortification You build the front wall of the prison with very strong fortification, therefore it requires no soldier to watch. Determine an arrangement of the minimum number of soldiers such there are at least 5 soldiers watching each wall (W1-W5). What you need to do: write down the optimization problem, solve it by MATLAB, show the solution.

Relaxation Describe what happened to the (optimal) solution when we relax the constraint on number of soldiers required to watch the front wall from 2 to 0.

Sol. Variable: $x_1, x_2, \dots, x_6$ be #soldiers in each tower. 1 point

Cost function: minimize sum of soldiers: $\sum_{i=1}^{6} x_i = \mathbf{1}^\top \mathbf{x}$ 1 point

The constraints: $x_i \in \mathbb{N}$ 1 point

$$\begin{aligned} x_1 + x_6 &\geq 2 && \text{Front wall constraint} \\ x_1 + x_2 &\geq 5 && \text{wall 1 constraint} \\ x_2 + x_3 &\geq 5 && \text{wall 2 constraint} \\ x_3 + x_4 &\geq 5 && \text{wall 3 constraint} \\ x_4 + x_5 &\geq 5 && \text{wall 4 constraint} \\ x_5 + x_6 &\geq 5 && \text{wall 5 constraint} \end{aligned} \qquad \text{(3 pts)}$$

For the second part of the problem, the front wall constraint is removed. (2 pt)

Sol part 1 $[2, 3, 5, 0, 5, 0]$ or [0,5,0,5,0,5](1 pt), part 2 $[5, 0, 5, 0, 5, 0]$ or [0,5,0,5,0,5] (1 pt)

Relaxation: any sounds argument 1 pt. Model answer: relaxation problem has solution value better than or equal to the original problem

2.2 Rental property decision problem (7 points)

You own a rental property and you rent out the property for 10 weeks (consecutively) in total. You have received 6 requests specifying which week each renter would like to start and how many weeks they want to stay (duration). The rate of renting the property is 100 dollar per week normally, but if anyone rents the property for at least 4 weeks, then they are charged 70 per week. Decide which requests to accept in order to maximize your income. Formulate this problem as an integer program. State clearly the decision variable(s), the cost function and the constraint(s). Solve the optimization problem.

person	start week	duration
A	1	1
B	1	4
C	3	2
D	5	3
E	7	2
F	9	1

Sol using BIP Decision variable: X_{ij} where $i \in [10], j \in [6]$ meaning renting week i to person j 1 pt

Cost function (2 pt): maximize

$$100X_{11}+70(X_{12}+X_{22}+X_{32}+X_{42})+100(X_{33}+X_{43})+100(X_{54}+X_{64}+X_{74})+100(X_{75}+X_{85})+100X_{96}$$

Constraints: $X_{ij} \in \{0,1\}$ (rent or not rent, binary decision) and $\sum_j X_{ij} \leq 1 \forall i$ (at most one occupancy each week) 2 pts

Sol using "Set Packing model"

$$\max\ 100x_A + 280x_B + 200x_C + 300x_D + 200x_E + 100x_F \text{ s.t. } \begin{array}{ll} x_A + x_B & \leq 1 \\ x_B + x_C & \leq 1 \\ x_D + x_E & \leq 1 \end{array},\ \mathbf{x} \in \{0,1\}^6$$

Both solutions give rent to A,C,D,F, total income = 700 2 pts

2.3 Solving IP (12 points)

Consider the following BIP:

$$\begin{array}{ll} \min & 5x_1 + 4x_2 + 6x_3 + 2x_4 + 4x_5 \\ \text{s.t.} & x_1 + x_2 \geq 1 \\ & x_2 + x_5 \geq 1 \\ & x_2 + x_3 + x_4 \geq 1 \\ & x_2 + x_3 + x_4 + x_5 \geq 1 \\ & x_i \in \{0,1\} \text{ for all } i \end{array}$$

1. Is (are) there any redundant constraint(s)? If yes, simplify the problem.
2. Is this IP solvable? If yes, solve this IP. If no, explain why.
3. Make it as simple as you can, make-up a location problem question in real-life that, when formulating such a problem will give raise to this IP. Draw the map of the location problem.

Sol 1. Yes: 2/3/4 (2)
2. Yes (1), $\mathbf{x}^* = [0,1,0,0,0]^\top$ (1), $f^* = 4$ (1).
3. Problem made-up (2), correct map and correct constraint (4)
Give points as long as no contradiction.
Example solution: city 2 is connected with all other cities and city 2 is located in the center. The facility constraint is that at least 1 facility in 1-2 and at least 1 facility in 2-3-4-5

3 Reformulation (30 points)

Note: You do not need to solve the problems here.

3.1 Reformulate the LP to (7 points)

Reformulate the problem

$$\begin{array}{ll} \min & 3x_1 - x_2 \\ \text{s.t.} & -x_1 + 6x_2 - x_3 + x_4 \geq -3 \\ & 7x_2 + x_4 = 5 \\ & x_3 + x_4 \leq 2 \\ & -1 \leq x_2 \\ & x_3 \leq 5 \\ & |x_4| \leq 2. \end{array}$$

to a LP in the form

$$1. \max_{\mathbf{y}} \ \mathbf{c}^\top \mathbf{y} \ \text{ s.t. } \ \mathbf{A}\mathbf{y} \leq \mathbf{b}, \mathbf{y} \geq \mathbf{0} \qquad 2. \max_{\mathbf{y}} \ \mathbf{c}^\top \mathbf{y} \ \text{ s.t. } \ \mathbf{A}\mathbf{y} = \mathbf{b}, \mathbf{y} \geq \mathbf{0}$$

Then solved these problem by using `linprog` in MATLAB. If the problem has a solution, show the solution. If the problem has no solution, explain why.

Sol 1 Important note: $\max_{\mathbf{y}} \ \mathbf{c}^\top \mathbf{y}$ s.t. $\mathbf{A}\mathbf{y} \leq \mathbf{b}, \mathbf{y} \geq \mathbf{0}$ is NOT standard form nor canonical form!

1. Min to max gives: $\max -3x_1 + x_2$
2. Replace 1st inequality by $x_1 - 6x_2 + x_3 - x_4 \leq 3$
3. Equality constraint replaced by two inequlities: $7x_2 + x_4 \leq 5, \quad -7x_2 - x_4 \leq -5$
4. x_1 is free so $x_1 = y_1 - y_2$ with $y_1, y_2 \geq 0$
5. x_2 has nonzero lower bound so $x_2 = y_3 - 1$ with $y_3 \geq 0$
6. x_3 with nonzero upper bound so $x_3 = 5 - y_4$ with $y_4 \geq 0$
7. x_4 bounded below and above so $x_4 = y_5 - 2$ with $0 \leq y_5 \leq 4$
8. All together gives

$$\begin{array}{ll} \max & \mathbf{c}^\top \mathbf{y}, \quad \mathbf{c}^\top = [-3, 3, 1, 0, 0], \mathbf{y} = [y_1, \dots, y_5] \\ \text{s.t.} & \mathbf{A}\mathbf{y} \leq \mathbf{b}, \mathbf{y} \geq \mathbf{0}, \ \text{with } \mathbf{A} = \begin{bmatrix} 1 & -1 & -6 & -1 & -1 \\ 0 & 0 & 7 & 0 & 1 \\ 0 & 0 & -7 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -10 \\ 14 \\ -14 \\ -1 \\ -4 \end{bmatrix} \end{array}$$

Optimal $\mathbf{y}$ unbounded (because y_2 can go ∞)

Solution for part 2: from $\mathbf{A}\mathbf{y} \leq \mathbf{b}$ simply add slack variables $\mathbf{s}$ to give $[\mathbf{A} \ \mathbf{I}] \begin{bmatrix} \mathbf{y} \\ \mathbf{s} \end{bmatrix} = \mathbf{b}$ Adding slack variables do not change the cost function so solution is also unbounded.

3.2 Reformulate the problems as LPs (9 points)

- Minimize $\|\mathbf{Ax}-\mathbf{b}\|_\infty$,
- Minimize $\|\mathbf{Ax}-\mathbf{b}\|_1$,
- Minimize $\|\mathbf{Ax}-\mathbf{b}\|_1$ subject to $\|\mathbf{x}\|_\infty \leq 1$

Sol 3 pts each:

- Minimize t s.t. $\mathbf{Ax}-\mathbf{b} \leq t\mathbf{1}, \quad -(\mathbf{Ax}-\mathbf{b}) \leq t\mathbf{1}$
- Minimize $\mathbf{1}^\top\mathbf{s}$ s.t. $\mathbf{Ax}-\mathbf{b} \leq \mathbf{s}, \quad -(\mathbf{Ax}-\mathbf{b}) \leq \mathbf{s}$
- Minimize $\mathbf{1}^\top\mathbf{y}$ s.t. $-\mathbf{y} \leq \mathbf{Ax}-\mathbf{b} \leq \mathbf{y}, \quad -\mathbf{1} \leq \mathbf{x} \leq \mathbf{1}$

3.3 Solve this problem (4 points)

Find the scalar x that is the minimizer of the problem

$$\min_x \left(\Big|4-\max\{1,x\}\Big| + \Big|3-\max\{2,x\}\Big| \right).$$

If the problem has exactly one solution, show the solution. If the problem has more than one solution, show all the solutions. If the problem has no solution, explain why.

Sol Replace max terms

$$\min_{x,y,z} |4-y|+|3-z| \text{ s.t. } y \geq x,\ y \geq 1,\ z \geq x,\ z \geq 2$$

Remove absolute signs

$$\min_{x,y,z,u,v} u+v \text{ s.t. } u \geq 0,\ v \geq 0,\ -u \leq 4-y \leq u,\ -v \leq 3-z \leq v,\ y \geq x,\ y \geq 1,\ z \geq x,\ z \geq 2,$$

Rearrange

$$\min \begin{bmatrix} 0\\0\\0\\1\\1 \end{bmatrix}^\top \begin{bmatrix} x\\y\\z\\u\\v \end{bmatrix} \quad \text{s.t.} \quad \begin{bmatrix} 0 & 1 & 0 & -1 & 0\\ 0 & -1 & 0 & -1 & 0\\ 0 & 0 & 1 & 0 & -1\\ 0 & 0 & -1 & 0 & -1\\ 1 & -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 1 & 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x\\y\\z\\u\\v \end{bmatrix} \leq \begin{bmatrix} 4\\-4\\3\\-3\\0\\-1\\0\\-2\\0\\0 \end{bmatrix}$$

Sol: $y=4, z=3$ so $x \in \{3,4\}$

3.4 Reformulate the IP as BIP / 0-1 IP (9 points)

Reformulate the following problem to BIP.

$$\begin{aligned} \min\quad & x_1 + x_2 + x_3 \\ \text{s.t.}\quad & x_1 + x_2 \le 10 \\ & x_2 + x_3 \le 10 \\ & x_1 + x_3 \le 10 \\ & 0 \le x_i \le 3, \quad i \in \{1,2,3\} \\ & x_i \text{ is integer} \end{aligned}$$

(Hint: 3 = 1 + 2).

Sol Let $x_1 = y_{11} + 2y_{12},\ x_2 = y_{21} + 2y_{22},\ x_3 = y_{31} + 2y_{32}$, so

x_1	$=0$	$=1$	$=2$	$=3$
y_{11}	0	1	0	1
y_{21}	0	0	1	1

Then we have

$$\begin{aligned} \min\quad & y_{11} + 2y_{12} + y_{21} + 2y_{22} + y_{31} + 2y_{32} \\ \text{s.t.}\quad & y_{11} + 2y_{12} + y_{21} + 2y_{22} \le 10 \\ & y_{21} + 2y_{22} + y_{31} + 2y_{32} \le 10 \\ & y_{11} + 2y_{12} + y_{31} + 2y_{32} \le 10 \\ & 0 \le y_{ij} \le 1, \quad \forall\ i,j \\ & y_{ij} \text{ is integer} \end{aligned}$$

3.5 Reformulate to QP (5 points)

Reformulate the following problem to QP in the form of $\min \mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{p}^\top \mathbf{x}$ s.t. $\mathbf{A}\mathbf{x} \le \mathbf{b}$.

$$\min \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \|\mathbf{C}\mathbf{x} - \mathbf{d}\|_1$$

$$\min \begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix}^\top \begin{bmatrix}\frac{1}{2}\mathbf{A}^\top\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} + \begin{bmatrix}-\mathbf{A}^\top\mathbf{b}\\ \mathbf{1}\end{bmatrix}^\top \begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} \quad \text{s.t.} \quad \begin{bmatrix}\mathbf{C} & -\mathbf{I}\\ -\mathbf{C} & -\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} \le \begin{bmatrix}\mathbf{d}\\ -\mathbf{d}\end{bmatrix}$$

4 Cpacitated Location Planning (30 points)

You are going to formulate and solve two facilities location planning problems. In these problems, the facilities have a maximum capacity of demand that the facility is able to serve.

The problem is presented by an undirected graph $G = (V, E)$, with V denotes the set of vertices and E denotes the set of edge. Here V has two subset : $V = \{I, J\}$, where the demand nodes are represented by a set of vertices $i \in I$, the possible locations of the facilities are given by the set of vertices $j \in J$, and edges $e_{i,j} \in E$ only exist between vertices from $i \in I$ to those in $j \in J$. Furthermore, we assign positive weights to the edges $d_{i,j} \ge 0$, which represents a distance between vertices i and j (Note that it is possible to have zero distance between a demand node and a possible facility location.). We assign positive weights to the demand nodes h_i for $i \in I$, and this represents the amount of demand at a particular node. Moreover, each facility has a maximum capacity C of demand that can be served.

4.1 A graph

Given $\mathbf{I} = \{1, 2, 3, 4, 5\}$, $\mathbf{J} = \{1, 2, 3\}$, $\mathbf{h} = \{1, 1, 1, 1, 1\}$, $C = 2$ and

$$\mathbf{E}\Big\{(1,1),(2,1),(3,1),(4,1),(5,1),(1,2),(2,2),(3,2),(3,3),(4,3),(5,3)\Big\}, \quad \mathbf{d} = \begin{bmatrix} 1 & 1 & * \\ \frac{1+\sqrt{5}}{2} & 0 & * \\ 0 & \frac{1+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & * & 1 \\ \frac{1+\sqrt{5}}{2} & * & 0 \end{bmatrix}$$

where $d_{ij} = *$ means people cannot go from i to j directly.

Draw the graph and label the demand node i and possible facility location j. What is the shape of this graph?

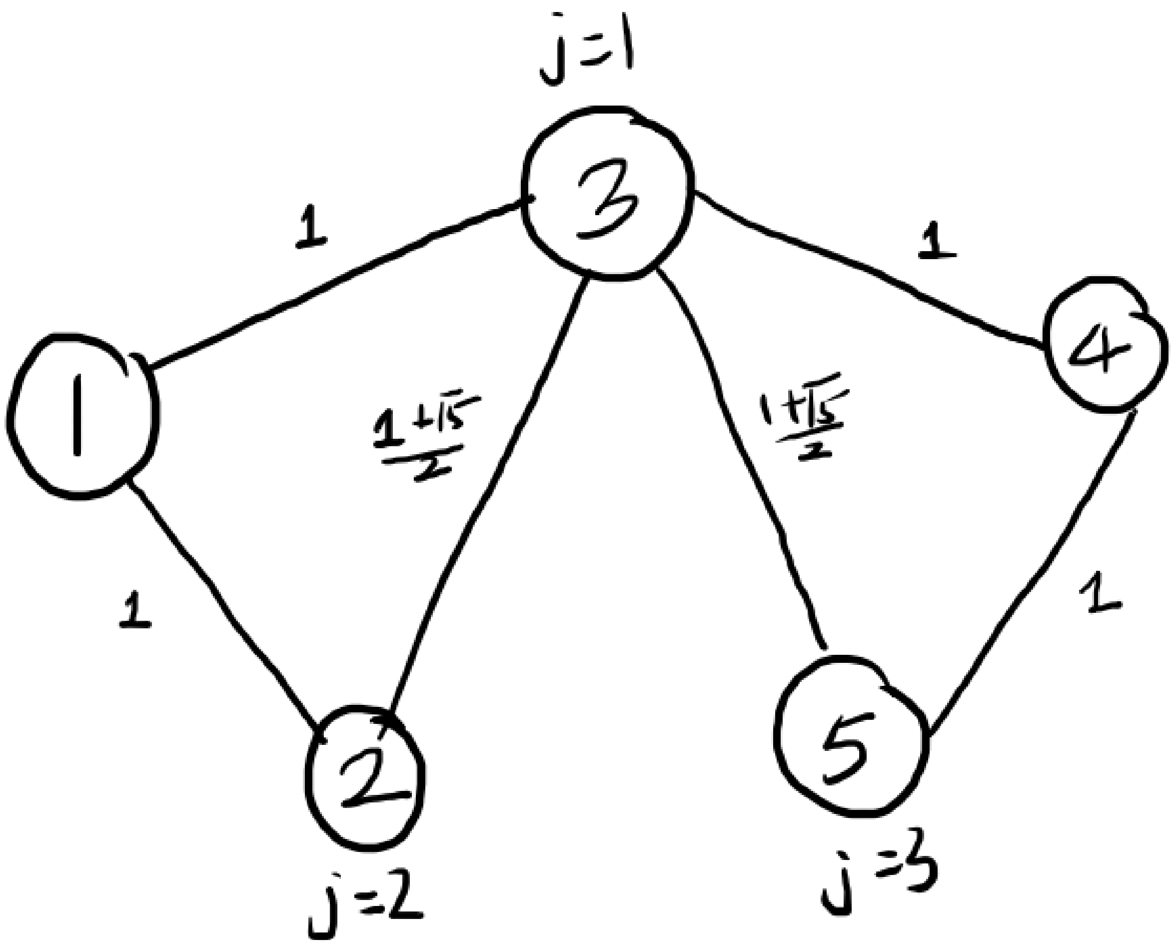


Figure 1: This worth 5 points. Any shape is ok as long as it is drawn correctly.

4.2 Capacitated p-Median Problem

In this problem, you wish to place p facilities to minimize the (demand-weighted) average distance between a demand node and the location in which a facility was placed, subject to the constraints on the capacity of the facilities. Formulate this capacitated p-median problem as the an ILP (Integer Linear Program). Solve this problem for $p = 1$.

Capacitated p-Median decision variables: x_{ij} and y_j

- x_{ij}: amount of demand at demand node i serviced by a facility placed at j. 1pt
- $y_j = 1$ if facility located at j and $= 0$ otherwise. (where a facility is placed) 1pt

min	$\sum_{j \in J} \sum_{i \in I} d_{ij} x_{ij}$	Cost function, 1pt
s.t.	$\sum_{j \in J} x_{ij} = h_i \quad \forall i \in I$	all node has all of its demand met, 1pt
	$x_{ij} \geq 0 \quad \forall i \in I, j \in J$	amount of demand served must be non-negative, 1pt
	$x_{ij} \leq h_i y_j \quad \forall i \in I, j \in J$	node can be serviced by a facility at j only if there is one, 1pt
	$\sum_{j \in J} y_j = p$	place exactly p facilities, 1pt
	$\sum_{i \in I} x_{ij} \leq C \quad \forall j \in J$	a facility at j can service at most C amount of demand, 1pt
	$y_j \in \{0, 1\} \quad \forall j \in J$	decision variables are binary, 1pt

Sol: No solution (we can place only 1 facility with max capacity of 2 and we need to serve 5 city with total demand 5), 1 pt

4.3 Capacitated p-Center problem

Now, you want to place p facilities to minimize the **maximum distance** between any demand node and its servicing facility, under the same constraints (capacity of the facilities.) Formulate this capacitated p-center problem as the an ILP. Solve this problem for $p = 1$.

Capacitated p-Center Decision variables: z, x, y, t

- x_{ij}: amount of demand at demand node i serviced by a facility placed at j. 1pt
- $y_j = 1$ if facility located at j and $= 0$ otherwise 1pt
- $w_{ij} = 1$ if demand node i is assigned to facility located at j and $= 0$ otherwise 1pt
- t: the max distance between any demand node and its servicing facilities. 1pt

min	t	minimize the max dist between any node and its serving facility
s.t.	$\sum_{j\in J} x_{ij} = h_i \quad \forall i \in I$	a demand node i must have all of its demand met.
	$x_{ij} \geq 0 \quad \forall i \in I, j \in J$	amount of demand served must be non-negative
	$x_{ij} \leq h_i y_j \quad \forall i \in I, j \in J$	demand i can be serviced by j only if there is one
	$x_{ij} \leq h_i w_{ij} \quad \forall i \in I, j \in J$	$z_{ij} > 0$ then y_{ij} is forced to be 1 since it is binary
	$w_{ij} \leq y_j \quad \forall i \in I, j \in J$	i can be served by j only if a facility is placed at j
	$\sum_{j\in J} y_j = p$	place exactly p facilities
	$\sum_{i\in I} x_{ij} \leq C \quad \forall j \in J$	a facility at j can service at most C amount of demand
	$t \geq d_{ij} w_{ij} \quad \forall i \in I, j \in J$	$t \geq$ the distance of any demand node and its servicing facility
	$y_j \in \{0,1\} \quad \forall j \in J$	decision variables are binary
	$w_{ij} \in \{0,1\} \quad \forall i \in I, j \in J$	decision variables are binary

Sol: no sol. 1 pt

- **Common mistake**: other than the new variable t that models the maximum, $w_{ij} \in \{0,1\}$ is needed because it links t and d_{ij} as $t \geq d_{ij} w_{ij}$. Here we cannot use $t \geq d_{ij} x_{ij}$ because x_{ij} is linear and it does not carry the binary information of "is i linked to j".
- Note: $t \geq d_{ij} w_{ij}$ will make sure if $x_{ij} = 0$ then $w_{ij} = 0$

5 Project management: Activity Scheduling (16 points)

You are going to formulate a complicated min-max problem. You are the manager of a big construction project.

- Consider a **project** that involves n **activities** (tasks, operations) to be performed **in parallel**, subject to a set of **temporal constraints**.
- The **start-finish** constraints specify the minimum time lag between the start of one activity and the finish of another. Each activity finishes as soon as its start-finish constraints are fulfilled.
- The **release time** and **release deadlines** define the earliest and latest allowed times (time window) for activities to start.
- The **flow-time** of an activity is the length of the time interval between its start and finish
- The **maximum deviation** of the start time is the difference between the latest and the earliest start times over all activities.
- The **scheduling objectives** are to minimize both the maximum flow-time and maximum deviation of the start time over all activities.

5.1 Modelling the start time and its deviation

Now, for each activity $i = 1, 2, \ldots, n$, let x_i be the unknown start time.

1. Write down the expression of the latest start time over all activities.
2. Write down the expression of the earliest start time over all activities.
3. Hence, write down the expression of the maximum deviation of the start time of activities.

$\max_{1 \le i \le n} x_i$ and $\min_{1 \le i \le n} x_i$. 2 pt (1 pt each)

$$\max_{1 \le i \le n} x_i - \min_{1 \le i \le n} x_i = \max_{1 \le i \le n} x_i + \max_{1 \le i \le n} -x_i \qquad \text{(2 pt)}$$

5.2 Modelling the flow-time

Now, together with the previously defined notation, let y_i be thee unknown finish time for each activity $i = 1, 2, \ldots, n$. Now

1. Write down the expression of the flow-time of activity i
2. Write down the expression of the maximum flow-time over all activity

1. $y_i - x_i$ 1 pt
2. $\max_{1 \le i \le n}(y_i - x_i)$ (if write down $\|\mathbf{y} - \mathbf{x}\|_\infty$ +1 bonus) 2 pt

5.3 Modelling the release time and release deadline constraints

Now, together with the previously defined notation, for each activity $i = 1, 2, \ldots, n$, let g_i be the given release time and h_i be the given relese deadline. Write down the release time and release deadline constraints

$$g_i \le x_i \le h_i, \quad i \in [n] \qquad \text{(2 pt)}$$

5.4 Modelling the start-finish constraints

Now, together with the previously defined notation, let a_{ij} be he given minimum possible time lag between the start of activity j and the finish of i. Write down the start-finish constraints.

$$\max_{1 \le j \le n}(a_{ij} + x_j) = y_i \qquad \text{(2 pt)}$$

5.5 Modelling the cost function

The scheduling objectives are to minimize both the maximum flow-time and maximum deviaation of the strt time over all activities. Write down the objective function.

$$\min_{x_i, y_i} \left\{ \max_{1 \le i \le n} (y_i - x_i) \ , \quad \max_{1 \le i \le n} x_i + \max_{1 \le i \le n} -x_i \right\} \qquad \text{(2 pt)}$$

5.6 Modelling the project scheduling problem

Write down the whole optimization problem.

$$\begin{array}{ll} \min\limits_{x_i, y_i} & \left\{ \max\limits_{1 \le i \le n} (y_i - x_i) \ , \quad \max\limits_{1 \le i \le n} x_i + \max\limits_{1 \le i \le n} -x_i \right\} \\ \text{s.t.} & \max\limits_{1 \le j \le n} (a_{ij} + x_j) = y_i \\ & g_i \le x_i \le h_i, \qquad i \in [n] \end{array} \qquad \text{(3 pt all correct, partly wrong : 2 or 1)}$$

6 Minesweeper (25 points)

Introduction (from wikipedia) Minesweeper is a single-player puzzle video game. The objective here is to clear a rectangular board containing hidden "mines" or bombs without detonating any of them, with the helps from the clues about the number of neighboring mines in each field. An example is shown in Fig.2.

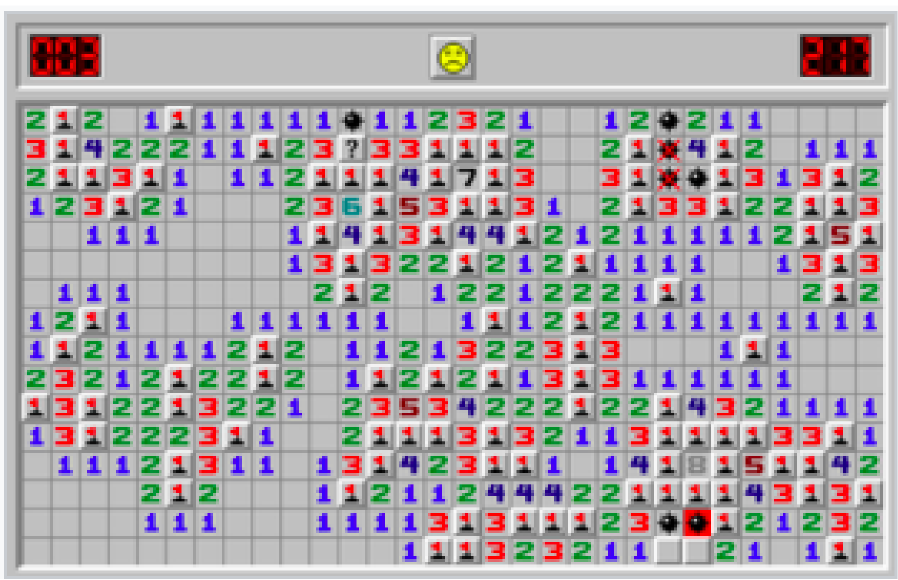
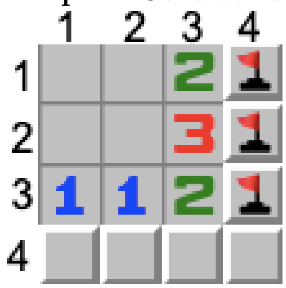


Figure 2: A typical minesweeper game with its commonly used style. This level (16 x 30 grid with 99 mines) is the "Expert" setting in typical implementations. Many boards have unavoidable guesses, causing lost games. Image from: wikipedia.

(If you have never played minesweeper, then before reading the following looooong chunk of words, the best to understand the mechanism of minesweeper is to try it yourself.)

The game description (Raphaël Collet): you are given a board represents a mine field, with mines hidden under the squares. Each square on the board may hide at most one mine. The total number of mines is known by the player. The game consists in finding the mines without making them explode. You get new hints each time you uncover a non-mined square. A move consists in uncovering a square. If the square holds a mine, the mine explodes and the game is over. Otherwise, a number in the square indicates how many mines are held in the surrounding squares, which are the adjacent squares in the eight directions north, north-east, east, south-east, south, southwest, west, and north-west. The goal of the game is to uncover all the squares that do not hold a mine.

A concrete example Consider a 4-by-4 mine field. In this example, there are 4 mines in the field.



$$F = \begin{bmatrix} 0 & 0 & 2 & ? \\ 0 & 0 & 3 & ? \\ 1 & 1 & 2 & ? \\ ? & ? & ? & ? \end{bmatrix} \quad X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

Refer to the figure, a 4×4 board is given and some squares are uncovered. Each square is identified by its coordinates $(row, column)$. We have the following:

1. The squares (1,1), (1,2), (2,1), (2,2) have already been played, and have no mine in their respective surrounding squares.
2. The squares (3,1), (3,2) have been played, too, and are surrounded by one mined square each.
3. The squares (1,3) and (3,3) each have two mines in their neighborhood.
4. The square (2,3) has three mines around it.

In this example, the player might deduce from (2,3) that (1,4), (2,4) and (3,4) are mined (the flag), and by (3,3) that (4,1), (4,3) and (4,4) are safe move and lastly by (3,1) and (3,2) that (4,1) is mined.

Such board can be represented mathematically as a matrix F. And the solution to the minesweeper problem can be represented by a binary matrix $X = x_{ij}$: value 1 means that the corresponding (i, j) square is mined, while 0 means a safe square. Solving this minesweeper problem corresponds to find the x_{ij} value subject to the constraints of the game.

6.1 The example

- In the example, the variable X is being solved by deduction using the information (the list 1,2,3,4) and the game rule (there are totally 4 mines in the field, the number of the square gives the hint of how many mines are held in the surrounding squares). By using these information, mathematically we uncover the unknown X as

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 0 & 0 & 0 & x_{14} \\ 0 & 0 & 0 & x_{24} \\ 0 & 0 & 0 & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \xrightarrow{2,3,4} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Now, write down mathematically the constraints 2,3,4 in the list.

$$\begin{aligned} x_{41} + x_{42} &= 1 && \text{The } 1 \text{ in } (3,1) \\ x_{41} + x_{42} + x_{43} &= 1 && \text{The } 1 \text{ in } (3,2) \\ x_{14} + x_{24} &= 2 && \text{The } 2 \text{ in } (1,3) \\ x_{24} + x_{34} + x_{42} + x_{43} + x_{44} &= 2 && \text{The } 2 \text{ in } (3,3) \\ x_{14} + x_{24} + x_{34} &= 3 && \text{The } 3 \text{ in } (2,3) \end{aligned}$$

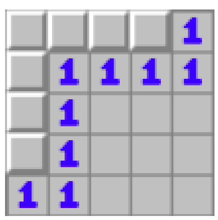
(5 pts)

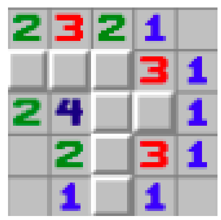
6.2 Constraint Satisfaction Problem

Such a problem is an example of *Constraint Satisfaction Problem* (CSP): there is there is no cost function. Mathematically,

minimize 0 subjec to constraints on x

Now, write down the full optimization problem for the following two minesweeper problems. Are these minesweeper problems solvable? If yes, solve these minesweeper problems and show the solution. If no, explain why they are not solvable.





$$\text{Sol:} \begin{bmatrix} M & & & M & 1 \\ & 1 & 1 & 1 & 1 \\ & 1 & & & \\ M & 1 & & & \\ 1 & 1 & & & \end{bmatrix} \text{5pt,} \qquad \begin{bmatrix} 2 & 3 & 2 & 1 & \\ M & M & M & 3 & 1 \\ 2 & 4 & M & M & 1 \\ & 2 & & 3 & 1 \\ & 1 & M & 1 & \end{bmatrix} \text{10pt}$$

Model 75%, solution 25%

6.3 More questions

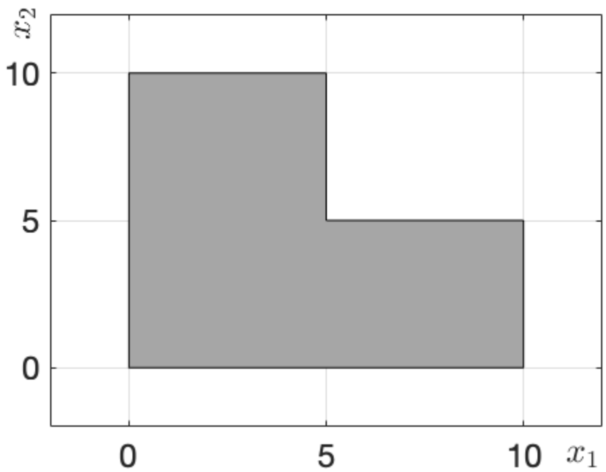
Solve the following minesweeper problem. You do not need to show the full optimization problem, you just need to show where is (are) the mine(s).

$$F = \begin{bmatrix} ? & 1 & ? & ? & ? \\ ? & ? & 3 & 3 & ? \\ 3 & ? & 4 & 2 & ? \\ ? & ? & ? & ? & 0 \\ ? & 2 & ? & ? & ? \end{bmatrix} \quad \text{Sol:} \begin{bmatrix} - & 1 & - & M & M \\ - & M & 3 & 3 & M \\ 3 & M & 4 & 2 & - \\ - & M & M & - & 0 \\ - & 2 & - & - & - \end{bmatrix} \text{5 pts}$$

7 Simple LP (10 points)

Consider a simple LP in $\mathbb{R}^2$ (2-dimensional space).

1. Draw the feasible region of $0 \leq x_1 \leq 10$, $0 \leq x_2 \leq 10$.
2. Write down the inequalities correspond to the feasible region of the figure in the right. (Hint: you need to introduce new variables).
3. Let $\mathbf{C}$ represents the feasible region of the figure in the right. Design a simple LP $\max \mathbf{c}^\top \mathbf{x}$ s.t. $\mathbf{x} \in \mathcal{C}$ such that when solving this LP, there are exactly TWO solutions (i.e., the solution is not unique and there are exactly two solutions).



sol

1. Trivial, 1 pt
2. $$\begin{aligned} 5 - x_1 + 5y_1 &\geq 0 \\ 5 - x_2 + 5y_2 &\geq 0 \\ y_1 + y_2 &= 1 \\ y_1, y_2 &\in \{0, 1\} \end{aligned} \qquad \text{(8 pts)}$$
   If using $x_3 = \min\{x_1, x_2\} \leq 5$, only get 5 to 6.
   If using "if" or similar cases condition get 2 or 4.
   If there is a contradiction get 1 to 0.
3. $\mathbf{c} = [1, 1]^\top$ (1 pt).

8 Marriage Matchmaking (15 points)

There are n men $i \in \{1, 2, \ldots, n\}$ and n women $j \in \{1, 2, \ldots, n\}$ do not want to be single anymore and all of them want to have a marriage. You are a professional matchmaker and your job is to match these people together.

- For simplicity, assume all these people are heterosexual (no gay nor lesbian), and each person has a preference list of the opposite sex.
- Suppose M_{ij} is the score of woman j given by man i, and W_{ij} is the score of man i given by woman j. Here, a higher score means a higher preference.
- A marriage, is a one-to-one correspondence between a man and a woman (that is, we do not consider the case of man-man-woman nor woman-woman-man marriage).
- An unstable marriage is that if there exist two marriage couples (i_1, j_1) and (i_2, j_2) such that man i_1 desires j_2 more than j_1 and woman j_1 desires i_2 more than i_1. In this case (i_1, j_1) is an unhappy couple.

The questions

1. Given the matrices M and W, describe how you will solve such marriage matchmaking problem (matching couples such that there is no unstable marriage) by formulating an optimization problem. Describe what are the decision variables, what is the objective function, what are the constraints, and what kind of optimization problem it is.

2. Now, consider the cases that some people are gay / lesbian / bisexual. Describe how you will solve the marriage matchmaking problem (matching couples such that there is no unstable marriage), describe how you will reformulate the optimization problem in part 1 in this new case.

3. Now, suppose you are in a place where bigamy (multiple marriage in the form of man-man-woman or man-woman-woman) is allowed. Describe how you will solve the marriage matchmaking problem (matching couples such that there is no unstable marriage), describe how you will reformulate the optimization problem in part 1 in this new case.

Sol Part 1 (6 pts in total) Decision variable : $\mathbf{X} = x_{ij}$, pair man i to woman j 1 pt

max	$\sum_{i,j}(M_{ij} + W_{ij})x_{ij}$	1 pt
s.t.	$\mathbf{X} \in \{0,1\}^{n\times n}$	binary decision, 1 pt
	$\mathbf{X1} = \mathbf{1}$	**X**each man pair with exactly 1 woman, 1pt
	$\mathbf{X}^\top\mathbf{1} = \mathbf{1}$	**X**each woman pair with exactly 1 man, 1pt
	$(M_{aj} + W_{ib})x_{ij} \geq (M_{ij} + W_{ij})x_{ab}$	stability of couple ij, 1pt

Remarks

- For cost function, $M_{ij}W_{ij}$ is not correct
- For the stability, also accept $(M_{aj} + W_{ib} - (M_{ij} + W_{ij}))x_{ab}x_{ij}$ this time.
  If $(M_{aj} + W_{ib} - (M_{ij} + W_{ij})) \geq 0$ only get 0.5pt
- It is possible to model the stability constraint as a linear constraint, in this case give +2 bonus if correct

Part 2 (5 pt in total) Now we do not need distinction for man and woman:

- We replace M, W by one single large matrix Q_{ij} to represent the preference of all people 2pt
  If student answer says "add new matrix" then only get 1 pt
- x_{ij} has a new additional constraint that $x_{ii} = 0$ (prevent being single) 2pt
  Or mention $Q_{ii} = 0$
  Or mention you cannot marry yourself
  Or mention anything with the meaning "prevent being sigle"
- Other parts similar as part 1 (mention "same approach" / "same formulation") 1 pt

Part 3 (4 pt in total)

- 4 pt if mention using Q_{ijk} / x_{ijk} / 3D decision variables, otherwise:
- 3 pt if few mistakes
- 2 pt if wrong but make sense
- 1 pt if mostly wrong
- 0 pt for nonsense

9 About the course (10 points)

Please answer the following question honestly. This will help improve this course.

- What do you think about the difficulty of the course so far? For example, you think the difficulty of the assignments is (Choose one):

  too easy, easy, acceptable, hard, too hard

- What do you think about the work load of assignments so far? Choose one:

  want more assignment, light, acceptable, little heavy, too many assignments

- You might feel that the questions in the assignments / lecture material are not like those textbook questions you typically see in other courses. Compared with other courses, what do you think about these type of questions / material used in this course? Choose one

  very interesting, more fun than usual, no comment, boring as usual, very boring

- Name one thing you like about the course.
- Name one thing you don't like about the course.

-END-