L_p norms : ℓ_1, ℓ_∞ and ℓ_2 norm

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What is norm

- "Length"
- Definition $\ell(x)$ is a norm if

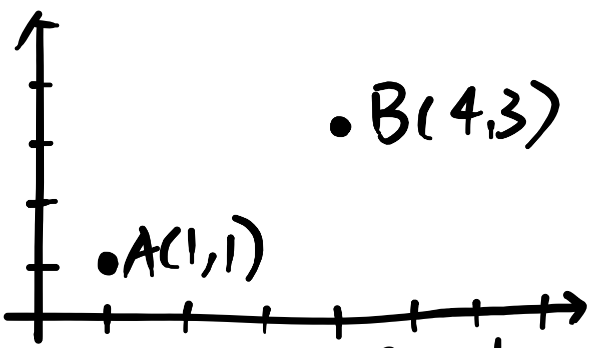
1. $\ell(x) = 0 \Leftrightarrow x = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$
2. $\ell(\alpha x) = |\alpha| \, \ell(x)$
3. $\ell(x+y) \leq \ell(x) + \ell(y)$

- length of x : $\ell(x) = \ell(x-0)$
- length of line segment x, y : $\ell(x-y) = \ell(y-x)$

ℓ_p-norms

- $\ell_p(x) = \|x\|_p \triangleq \left(\sum_i x_i^p\right)^{\frac{1}{p}}$
- $\ell_2(x) = \|x\|_2 = \sqrt{\sum_i x_i^2} = \sqrt{x_1^2 + x_2^2 + \cdots}$
- $\ell_1(x) = \|x\|_1 = \sum_i |x_i| = |x_1| + |x_2| + \cdots$
- $\ell_\infty(x) = \|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_n|\}$
- $|\cdot|$ is absolute value

Examples



	ℓ_1	ℓ_2	ℓ_∞
A	2	$\sqrt{2}$	1
B	7	5	4
A-B	5	$\sqrt{13}$	3

Norm minimization with linear constants

$$\min_x \|x\|_p \quad (\text{or } \|Ex\|_p)$$
$$\text{s.t. } Ax = b$$
$$x \geq 0$$

- For ℓ_1, ℓ_∞ norms, this problem can be re-formulated as an LP!
- For ℓ_2 norm, this problem is a QP!

Example on ℓ_2 norm

$$\min_x \|x\|_2 \iff \min_x \|x\|_2^2$$
$$\iff \min_x x^T x$$
$$\iff \min_x x^T Q x, \quad Q = I$$

$\Rightarrow$ can be solve by QP solver

- What about $\min_x \|Ex\|_2$?

Example: ℓ_∞ case

- $\min_x \|x\|_\infty \iff \min_{x_1, x_2, \dots} \max\{|x_1|, |x_2|, \dots\}$

- By the extreme-case trick (Lecture 3)
  Let $t \geq 0$ such that $|x_i| \leq t \ \forall i$

- We have $\left.\begin{aligned} \min\ & t \\ \text{s.t. }\ & |x_i| \leq t \\ & t \geq 0 \end{aligned}\right\}$ a LP!

Example

- Original problem $\min_x \|Ax - b\|_\infty$

- LP re-formulation:

$$\begin{aligned} \min\ & t \quad \leftarrow \text{scalar} \\ \text{s.t. }\ & Ax - t\mathbb{1} \leq b \quad \leftarrow \text{vector of } 1 \\ & -Ax - t\mathbb{1} \leq -b \\ & t \geq 0 \end{aligned}$$

Example : ℓ_1 case

- $\min_x \|x\|_1 = \min \sum_i |x_i|$ ← not linear!

- Trick : introduce $u_i \geq 0$ such that

$$|x_i| \leq u_i \quad \text{for all } i$$

- We have:

$$\min \sum u_i$$ ← linear!

$$\text{s.t. } |x_i| \leq u_i$$ ← Not linear!

$$u_i \geq 0$$ ← linear!

Absolute value

- $|x| \leq a \iff -a \leq x \leq a \iff \begin{matrix} x \leq a \\ -x \leq a \end{matrix}$

- LP re-formulation of ℓ_1 problem

$$\left.\begin{aligned} \min\ & \mathbb{1}^T u \\ \text{s.t. }\ & x \leq u \\ & -x \leq u \\ & u \geq 0 \end{aligned}\right\} \text{LP!}$$

Example: $f(Ax)$

- Oringinal problem $\min \|Ex\|_1$ s.t. $Ax = b$, $x \geq 0$

$$\begin{aligned} \min\ & \|Ex\|_1 \\ \text{s.t.}\ & Ax = b \\ & x \geq 0 \end{aligned}$$

- Possible trick: introduce $y = Ex$

$$\begin{aligned} \min\ & \|y\|_1 \\ \text{s.t.}\ & Ex = y \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

and go on...

Application of ℓ_1, ℓ_∞ norms

- Robust curve fitting
- Image inpainting
- Linear Classification

... and any problem that can be modeled by ℓ_1 or ℓ_p norm.