Extra examples.

1. Solve

$$\min_{x} \left| 4 - \max(6, 2x) \right| \quad \cdots \quad (P)$$

Step 1. Remove the max

Introduce $y = \max(6, 2x) \Rightarrow \begin{cases} y \geqslant 6 \\ y \geqslant 2x \end{cases}$

Now P becomes

$$\min_{x,y} \left| 4 - y \right| \quad \text{s.t.} \quad \begin{array}{l} y \geqslant 6 \\ y \geqslant 2x \end{array} \qquad (P_1)$$

Step2. Remove the absolute value

Note, we are minimizing $|4-y|$, so to remove $|\cdot|$,

we introduce an upper bound

$$|4-y| \leq u \Rightarrow -u \leq 4-y \leq u \Rightarrow \begin{cases} -u \leq 4-y \\ 4-y \leq u \end{cases}$$

Now P_1 becomes

$$\min_{x,y,u} \ u \quad \text{s.t.} \quad \begin{aligned} & y \geq 6 \\ & y \geq 2x \\ & -u \leq 4-y \\ & 4-y \leq u \end{aligned} \qquad P_2$$

Step3. Convert P_2 to Canonical Form

$$P_2: \quad \min_{x,y,u} \; u \qquad \Longrightarrow \qquad \min_{x,y,u} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ y \\ u \end{bmatrix}$$

$$\text{s.t.} \quad \begin{aligned} y &\geq 6 & &\rightarrow & -y &\leq -6 \\ y &\geq 2x & &\rightarrow & 2x - y &\leq 0 \\ -u &\leq 4 - y & &\rightarrow & y - u &\leq 4 \\ 4 - y &\leq u & &\rightarrow & -y - u &\leq -4 \end{aligned}$$

$$\text{s.t.} \quad \begin{bmatrix} 0 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix} \leq \begin{bmatrix} -6 \\ 0 \\ 4 \\ -4 \end{bmatrix}$$

Step 4. Call MATLAB

```
clear; close all;clc
c = [0; 0; 1]
A = [0 -1  0; ...
     2 -1  0; ...
     0  1 -1; ...
     0 -1 -1]
b = [-6; 0; 4; -4]
[x, fval] = linprog(c,A,b)
```

```
Command Window
c =

     0
     0
     1


A =

     0    -1     0
     2    -1     0
     0     1    -1
     0    -1    -1


b =

    -6
     0
     4
    -4


Optimal solution found.


x =

     3
     6
     2


fval =

     2
```

$$\begin{pmatrix} x \\ y \\ u \end{pmatrix} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

So $x=3$ in $\min\limits_{x} \left| 4 - \max(6, 2x) \right|$

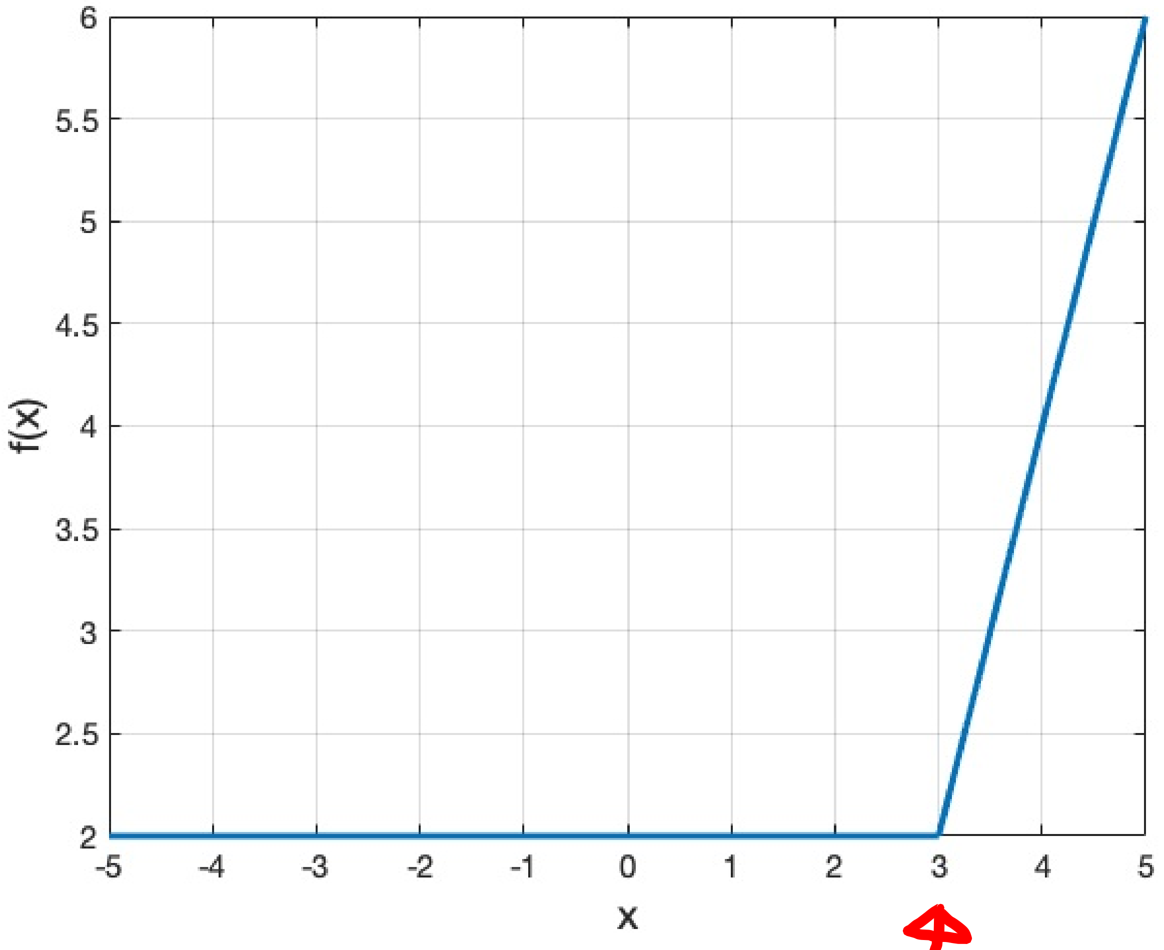
If we double check with brute force

```
clear; close all ;clc
% creat x as a vector from -5 to 5
x = -5:0.01:5;

% compute the function value
f = abs(  4 - max( 6, 2*x) );

% Plot the result
plot(x,f, 'linewidth', 2)
xlabel('x','fontsize',20)
ylabel('f(x)','fontsize',20)
grid on, axis tight
```

$\longrightarrow$



$x=3$ is really a solution

☆ $\min |4-\max(6,2x)|$ has ~~5~~ many solutions!

Practise problem

$$\max_{x_1, x_2} \quad 2\|Ax\|_1 + 3\|Bx\|_\infty$$

$$\text{s.t.} \quad \mathbb{1}^T x \le 10$$

where $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $\mathbb{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Sol.

$$\max_{x_1,x_2} \; 2\|Ax\|_1 + 3\|Bx\|_\infty$$
$$\text{s.t.} \quad \mathbb{1}^T x \le 10$$
$$\text{where} \quad A=\begin{bmatrix}1&0\\0&0\end{bmatrix}, \quad B=\begin{bmatrix}1&0\\0&1\end{bmatrix}, \quad \mathbb{1}=\begin{bmatrix}1\\1\end{bmatrix}$$

$$\Rightarrow \quad \max \; 2\left\|\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|_1 + 3\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|_\infty$$
$$\text{s.t.} \quad x_1 + x_2 \le 10$$

$$\Downarrow$$

$$\max \left(2|x_1| + 3\max\{x_1, x_2\}\right)$$
$$x_1 + x_2 \le 10$$

Sol $\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}+\infty\\-\infty\end{pmatrix}$

sol at ∞!

$$\Downarrow$$

$$\max \; l + 3\max\{x_1, x_2\}$$
$$x_1 + x_2 \le 10$$
$$l \le 2|x_1|$$

* Here the problem is MAX so we introduce LOWER BOUND l instead of upperbound

$$\Leftarrow \quad \max \; l + y$$
$$\text{s.t.} \quad x_1 + x_2 \le 10$$
$$l \le 2|x_1|$$
$$y \ge x_1$$
$$y \ge x_2$$

$$\Leftarrow \quad \max \begin{pmatrix}0\\0\\1\\1\end{pmatrix}^T \begin{pmatrix}x_1\\x_2\\l\\y\end{pmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix}1&1&0&0\\-2&0&1&0\\2&0&1&0\\1&0&0&-1\\0&1&0&-1\end{bmatrix}\begin{pmatrix}\;\end{pmatrix} \le \begin{bmatrix}10\\0\\0\\0\\0\end{bmatrix}$$

Summary

$$\max \; |f(x)| \iff \max \; l$$

$$l \le |f(x)| \iff \begin{array}{l} l \le f(x) \\ f(x) \le -l \end{array}$$

$$\min \; |f(x)| \iff \min \; u$$

$$u \ge |f(x)| \iff -u \le f(x) \le u$$

And the extreme-case trick (previous lecture)

$$\min \; \max\{f_1(x), f_2(x), \dots\} \iff \begin{array}{ll} \min & y \\ \text{s.t.} & y \ge f_1(x), \; y \ge f_2(x), \; \dots \end{array}$$

END